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Volume I

Analytical Methods

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Volume I - Analytical Methods

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Contents

Preface

This book is mainly based on the material initially published in Serbian, in 2021, by the University of Belgrade, Faculty of Mining and Geology, under the title Mathematical Physics (Theory and Examples). For the purpose of this book the material from the Serbian edition was reviewed, amended, and translated, with new material added in two final chapters in the second volume. We have divided text into two separate volumes:

Mathematics of Physics - Analytical Methods and
Mathematics of Physics - Numerical Methods.

The first volume consists of 8 chapters:

- The first 7 chapters were written by Dragoslav Kuzmanović, Dobrica Nikolić and Ivan Obradović, and correspond to the text from Chapters 1-8 of the Serbian edition, translated by Ivan Obradović.
- The material of Chapter 8, which is of a monographic character, corresponds to the material of Chapter 9 in the Serbian edition, but was thoroughly reviewed and rewritten in English by Mihailo Lazarević.

The second volume consists of 6 chapters:

- The first 3 chapters were written by Aleksandar Sedmak and correspond to Chapter 10 of the Serbian edition, restructured and reviewed, and then translated by Simon Sedmak.
- Chapter 4 corresponds to the text of Chapter 11 of the Serbian edition, written and translated by Nikola Mladenović.
- Chapters 5 and 6, written by Rade Vignjević and Sreten Mastilović, respectively, offer completely new material.

Chapters 4, 5 and 6 are of a monographic character.

I Vector algebra and analysis

1. Vector algebra

1.1 Introduction - On scalars, vectors and tensors

We encounter various phenomena in the space that surrounds us and define the concepts that characterize them in order to describe them. However, it has been noted that different phenomena can, mathematically, be described in the same way, that is, they can be elements of the same set in which certain mathematical rules apply. Quantities such as: length, area, volume, mass, temperature, pressure, or electric charge can be specified by a single number (namely, the number of units of a conveniently chosen measurement scale, such as: $3m$, $0.5m^2$, $10°C$, $1bar$, $110V$, etc.). These quantities are called **scalars**. The choice of scale is a matter of agreement and depends on practical problems (practical needs).

However, we also encounter (physical) quantities that require more data (parameters) in order to be defined. Examples of such quantities are: movement of a point, speed, acceleration, force, etc. These quantities are characterized by direction and magnitude, and we call them **vectors**.

Finally, there are quantities that require even more parameters in order to be defined. Thus, for example, inertia, which captures the relation between angular velocity and angular momentum for a rigid body, is determined by nine independent data (components). Such quantities, if they follow specific physical laws, are called **tensors**.

In this chapter we will study vectors. However, before we define vectors and relevant operations, we will define the coordinate system, since we will later need it to work with vectors more conveniently.

1.2 Coordinate system

In order to determine the position of geometric objects, it is necessary to define the reference system in relation to which they are observed.

The basic idea (Descartes)[1] is to assign a unique n-tuple of numbers to each point in the

[1]René Descartes (Latin name Renatus Cartesius) (1596-1650), French philosopher and mathematician. He introduced analytical geometry. His seminal work *Géométrie* appeared in 1637, as an addition to his work *Discours de la méthode*.

n-dimensional space.

Thus, in a real one-dimensional space (which is geometrically represented by a straight line), to each point a real number is assigned, whose absolute value is the distance (we will define the general term “distance” later) from a predetermined point, for example O, called the origin of the coordinate system. In addition to the origin, it is necessary to determine the unit of distance (the distance to which all other distances shall be compared). To that end, a point A is selected, and the distance $\overline{OA}$ is considered to be the unit distance. Let P be an arbitrary point, then the number x, assigned to the point P, is defined as follows

$$|x| = \frac{\overline{OP}}{\overline{OA}}. \tag{1.1}$$

If the point is to the right of point O (in our example the point P in Figure 1.1), a plus sign (+) is assumed, namely $x > 0$, and if it is to the left (in our example the point Q in Figure 1.1) than the sign (–) is assumed, namely $x < 0$.

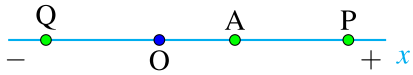
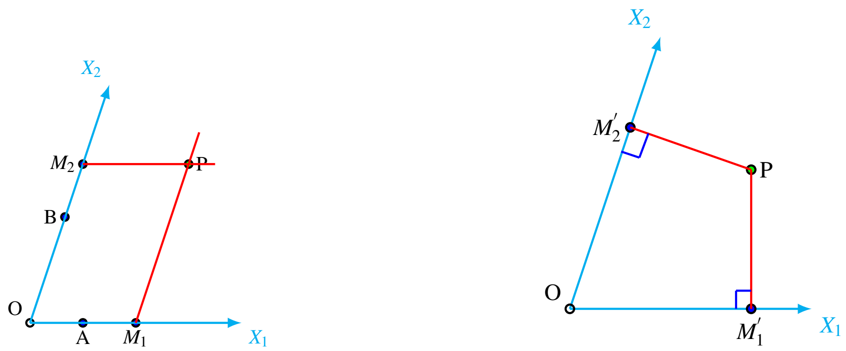


Figure 1.1: Oriented straight line.

In this way we determine the direction of the "movement" of a point, and an oriented straight line called the **axis** is obtained. This orientation is denoted by an arrow indicating the direction in which the numbers are growing.

In the real two-dimensional space an ordered pair of real numbers is assigned to each point, with respect to two corresponding lines X_1 and X_2 that intersect at point O (Fig. 1.2). This point is called the **origin of the coordinate system**.



(a) (b)

Figure 1.2: Two ways for determining the position of a point.

In this it is also necessary to define a unit of distance, for each axis separately, which means that these distances do not necessarily have to be the same.

The pair of these axes, with units of distance $\overline{OA}$ and $\overline{OB}$, represents **axes of the coordinate system** in the plane.

To each point P in the plane an ordered par of real numbers (x_1, x_2) is assigned, which are called the **coordinates** of that **point**, and which are determined as follows. The straight line, which passes through point P, and is parallel to the X_2–axis, intersects the X_1–axis at point M_1, while the straight line parallel to the X_1–axis, intersects the X_2–axis at point M_2 (Fig. 1.2(a)).

Coordinates x_1 i x_2 are defined by:

$$|x_1| = \frac{\overline{OM}_1}{\overline{OA}}, \quad |x_2| = \frac{\overline{OM}_2}{\overline{OB}},$$

where the sign for x_1 and x_2 is determined in the same way as in the one-dimensional space.

By this procedure, an ordered pair of numbers (x_1, x_2) can uniquely be assigned to each point P from the plane (with respect to the given coordinate axes), thus defining the coordinate system of two-dimensional space.

This procedure can be generalized and applied to the n–dimensional space ($n > 2$).

If the angle between the straight axes is 90°, then such a coordinate system is called **Cartesian coordinate system** or rectangular (orthogonal) coordinate system.

R Note that the procedure for assigning an ordered pair of numbers to a point described above is not the only one used. Namely, it is also possible to draw straight lines from point P that are perpendicular to the corresponding axes (Fig. 1.2(b)), thus obtaining points M_1' i M_2'. In that case the point P has coordinates x_1' and x_2', defined by:

$$|x_1'| = \frac{\overline{OM'}_1}{\overline{OA}}, \quad |x_2'| = \frac{\overline{OM'}_2}{\overline{OB}}.$$

In the special case of Cartesian coordinate system the pairs of numbers (x_1, x_2) and (x_1', x_2') are the same. In addition to these procedures for assigning coordinates other procedures are also possible, but these two are generally used in practice.

In the previous definitions, the term distance was used, which has so far not been defined. It should be noted that, depending on the expression that defines the distance between two points, different spaces (in mathematical terms) can be distinguished. Thus, for example, the distance between two points A, with Cartesian coordinates (a_1, a_2) and B, with Cartesian coordinates (b_1, b_2), can be defined by the expression

$$d_{AB} = \sqrt{\sum_{i=1}^{2} (b_i - a_i)^2} \equiv \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}. \tag{1.2}$$

In the n–dimensional space this distance is given by the expression

$$d_{AB} = \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}. \tag{1.3}$$

1.3 Vector algebra

In the previous section, construction of a coordinate system in two-dimensional space, which is intuitively close to human perception, was reviewed. In this system the distance between two points is measured by Pythagoras[2] formula (1.3). If, in such a space, a point is moved from position A to a new position B, this movement from A (start point) to B (end point) can be represented by the oriented straight line segment $\overrightarrow{AB}$ (Fig. 1.3).

[2] $\Pi\upsilon\vartheta\alpha\gamma o\rho\alpha\varsigma$, Greek philosopher and mathematician. Born around 570 B.C. and died around 497 B.C. He is considered the founder of theoretical mathematics and research in physics (acoustics).

Definition

An oriented straight line segment is called a **vector**. The length of the segment is the **magnitude** of the vector.

The vector[3] defined in this way represents a geometric concept, as opposed to the previous definition (movement), which gave the vector a physical meaning.

It is common in the literature to denote a vector by one letter in bold (**a**)[4] or by $\overrightarrow{AB}$ (A is the start, and B the end point) when it is important to emphasize the start and end points. In this book, both ways of denoting vectors will be used equally.

1.4 Operations on vectors

1.4.1 Addition of vectors

Consider moving a point from position A to position C. Position C can be reached directly or via position B. This operation can be denoted by the following relation (Fig. 1.3)

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}. \tag{1.4}$$

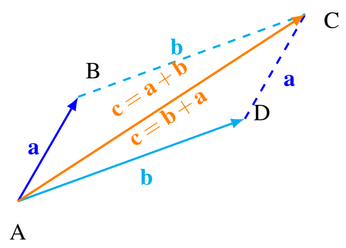


Figure 1.3: Addition of vector.

If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$, the previous operation can be represented in one of the following ways:

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \quad \text{or} \quad \vec{a} + \vec{b} = \vec{c} \quad \text{or} \quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}. \tag{1.5}$$

The vector composition rule was first formulated by Stevinus [5] in 1586, within his studies of force composition laws (Fig. 1.4).

[3] The origin of this term comes from the Latin word *vector* – carrier, or from *vehere, vectum* – to carry, to move.

[4] The boldface letter is common in printed materials. However, as it cannot be used in handwriting, an arrow over the letter is used instead, e.g. $\vec{a}$, instead of **a**. In the case where the vector is determined by the start point A and end point B the notation $\overrightarrow{AB}$ i used.

[5] Stevin Simon - Stevinus (1548-1620), Dutch mathematician and physicist. He was one of the first to use experiments in his research. He was also the first to define the law on the balance of forces on a steep plane and formulate the law of parallelogram of forces. His notable works are in fluid mechanics.

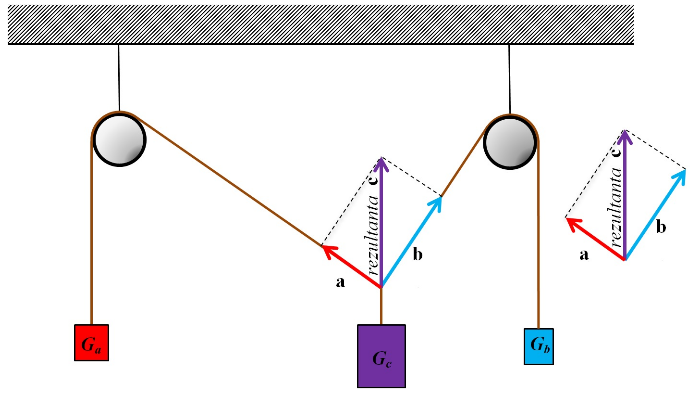


Figure 1.4: Sum of vectors as equilibrium of forces.

In literature, this rule is known as the **parallelogram law** of addition, as (see Figures 1.3 and 1.4) the sum of vectors **a** and **b** is represented by the diagonal of the parallelogram *ABCD*. Addition of vectors is, thus, a **binary operation** over a set of vectors $\mathbb{V}$, by which a vector **c** is uniquely assigned to vectors $\mathbf{a},\mathbf{b} \in \mathbb{V}$.

The fact that many quantities in physics can be represented by oriented straight line segments, which are summed according to the parallelograms law, prompts the study of vectors in more depth. Thus, by introducing vectors, physical quantities are geometrized.

R Note that there are situations in physics in which it is necessary to impose a boundary on the start point or position of the line - carrier of the observed vector. Two examples (**rigid** and **deformable body**)[6] follow.

Example 1

Let us observe a **rigid body**. One of the axioms of statics is: two systems of forces are statically equivalent if the difference between them amounts to a system of forces in equilibrium.

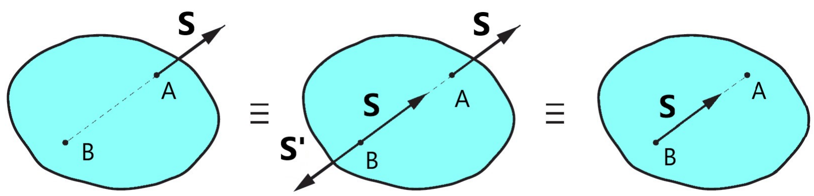


Figure 1.5: Movement of the force - rigid body

An important consequence of this axiom is: the point of application of the force on a rigid body can move along the line of action of the force. Namely, if a system in equilibrium $(\vec{S'}, \vec{S})$ is added in point B (on the line of action of the force) (Fig. 1.5), and then the system in equilibrium ($\vec{S'}$ – point of application B, $\vec{S}$ – point of application A) is removed, then the force $\vec{S}$ still remains, but with the point of application B.

However, if the body is viewed as deformable, it is irrelevant at which point of the body the force is going to be applied.

[6] A body in which the distance between any two points does not change during its movement.

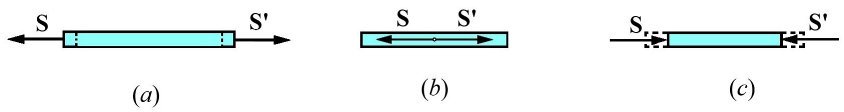


Figure 1.6: Force displacement - deformable body

For example, in Fig. 1.6*a* the rod is strained to tension, and the rod elongates. If the points of application of both forces move, say, to the center of the rod, Fig. 1.6*b*, the rod will not be strained. Finally, if forces are applied to opposite ends of the rod, then the rod will be strained to pressure (Fig. 1.6*c*), and the rod is shortened. Thus, from the standpoint of movement or resting of the rod, it is completely irrelevant whether it is affected by forces as in Figures 1.6. All three cases are equivalent. But from the standpoint of determining the internal forces in individual sections of the rod, the difference is essential.

The following vectors can be distinguished:

- **free** (they move parallel to themselves, but do not change; an example for this type of vector is the coupling momentum, the translation vector),
- **sliding** or vector bound to a line (it does not change when moving along the carrier line; for example, the force acting on a rigid body) and
- **bound** to a point (for example, volume forces).

R Note that the operations to be defined will only apply to free vectors, unless otherwise noted.

Starting from the idea of vectors as point displacements, we conclude that two vectors are equal if the oriented segments representing them are equal in length (equal in magnitude), and their directions are the same. We will denote this by

$$\mathbf{a}=\mathbf{b}. \tag{1.6}$$

Fig. 1.7 shows vector pairs that are not equal because they differ in magnitude (Fig. 1.7(a)) or direction (Fig. 1.7(b) and 1.7(c)).

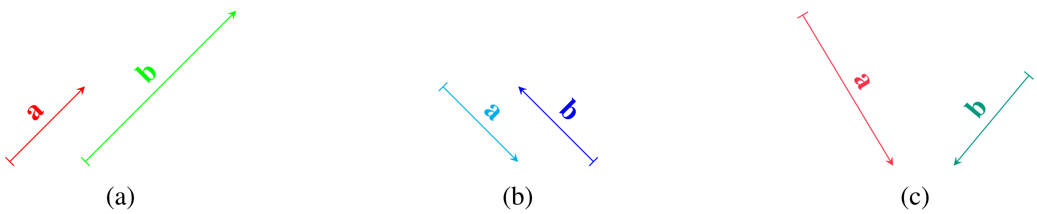


Figure 1.7: Vectors that differ (a) in magnitude (b) and (c) in direction.

We will denote the length (magnitude) of the vector **a** by $|\mathbf{a}|$ or shortly a.

Definition

Zero vector is a vector with zero displacement (a vector whose beginning and end coincide), and we denote it by $\mathbf{0}$. For each vector $\mathbf{a}$

$$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}. \tag{1.7}$$

The magnitude of zero vector is equal to zero and its direction is arbitrary (indefinite).

Definition

Two vectors of the same magnitude but opposite directions are called **opposite** vectors. The opposite vector to vector $\mathbf{a}$ is denoted by $-\mathbf{a}$. For these to vectors, the following applies

$$\mathbf{a}+(-\mathbf{a})=\mathbf{0}. \tag{1.8}$$

Definition

Each vector with a magnitude equal to one, i.e.

$$|\mathbf{a}|=1 \tag{1.9}$$

is called **unit vector**.

Based on the geometric properties of oriented segments, we conclude that:

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \quad (\textit{commutativity}) \tag{I}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \quad (\textit{associativity}) \tag{II}$$

Also note that the vector addition operation (+) is an internal[7] binary operation, i.e.:

$$\text{if} \quad \mathbf{a},\mathbf{b}\in\mathbb{V} \quad \text{then also} \quad \mathbf{a}+\mathbf{b}\in\mathbb{V}, \quad \text{where } \mathbb{V} \text{ is a vector set.} \tag{III}$$

Based on the previous definitions and properties, it can be briefly summarized that the following is true for the vector addition operation:

a) the operation is commutative (I),
b) the operation is associative (II),
c) the operation is internal (III),
d) the operation has a zero (neutral) element, $\mathbf{0}\in\mathbb{V}$ (1.7),
e) each element $\mathbf{a}\in\mathbb{V}$ has an opposite or symmetrical element $-\mathbf{a}\in\mathbb{V}$ for which

$$\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}. \tag{1.10}$$

A set $\mathbb{V}$, the elements of which have properties a) to e) in relation to an operation, is said to form a commutative or Abelian[8] group, or in other words, the set $\mathbb{V}$ has the structure of a commutative or Abelian group. Thus, based on the previous definition, it can be said that the **vector set $\mathbb{V}$ forms a commutative or Abelian group** with respect to addition.

Let us now define some more operations with vectors.

[7] An internal operation assigns to each element of a set an element from the same set.

[8] Niels Henrik Abel (1802-1829), Norwegian mathematician. He was the first to complete the proof demonstrating the impossibility of solving the general quintic equation in radicals. He also greatly contributed to the theory of elliptic functions and the theory of infinite series. He laid the foundation for the general theory of Abel integrals.

1.4.2 Multiplication of a vector by a real number (scalar)

Definition

Let **a** be a vector, and α a real number. Then $\alpha\mathbf{a}(\equiv \mathbf{a}\alpha)$ defines a new vector as follows:
- if $\mathbf{a} \neq 0$ and $\alpha > 0$, the new vector $\alpha\mathbf{a}$ has the same direction as vector **a**,
- if $\mathbf{a} \neq 0$ and $\alpha < 0$, the new vector $\alpha\mathbf{a}$ and the vector **a** have opposite directions,
- the magnitude of $\alpha\mathbf{a}$ is equal to $|\alpha\mathbf{a}| = |\alpha||\mathbf{a}|$ (if $\mathbf{a} = \mathbf{0}$ or $\alpha = 0$ (or both), then $\alpha\mathbf{a} = \mathbf{0}$).

It is said that the vector $\alpha\mathbf{a}$ is a result of the **multiplication** of the vector **a** by the scalar α.

We have thus defined the operation of multiplication of a vector by a real number (scalar).

The unit vector having the same direction as the vector **a** will be denoted as $\mathbf{e}_a$. Each vector can be represented using the operation of multiplication of a vector by a scalar as a product of its magnitude and its unit vector

$$\mathbf{a} = |\mathbf{a}|\mathbf{e}_a. \tag{1.11}$$

For the operation of multiplication of a vector by a scalar the following is true:

$$\alpha\mathbf{a} \in \mathbb{V}, \tag{IIIa}$$
$$(\alpha_1 + \alpha_2)\mathbf{a} = \alpha_1\mathbf{a} + \alpha_2\mathbf{a} \tag{IV}$$
$$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b} \tag{V}$$
$$\alpha_1(\alpha_2\mathbf{a}) = (\alpha_1\alpha_2)\mathbf{a}, \tag{VI}$$

for each real number α_1 and α_2 and each vector $\mathbf{a}, \mathbf{b} \in \mathbb{V}$.

The properties (IV–VI) are known as **linearity properties of the set** $\mathbb{V}$.

1.4.3 Projection on an axis and on a plane

Projection of a point on an axis

Consider an axis u determined by a unit vector **u**, a point A, which does not lie on that axis, and a plane S (Fig. 1.8), which is not parallel to the axis.

Construct a plane S' that contains point A and is parallel to plane S. The point A' at which the axis u intersects the plane S' is the **projection of the point** A on the axis u parallel to the plane S. If the plane S is normal to the axis, then the corresponding projection is called **normal** or **orthogonal**.

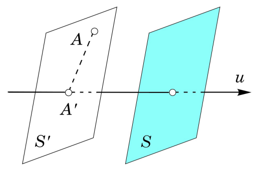


Figure 1.8: Projection of a point on an axis.

Projection of a vector on an axis

Let a vector be determined by its start point A and its end point B. By projecting these two points (Fig. 1.9), points A' and B' are obtained, that is, vector $\overrightarrow{A'B'}$.

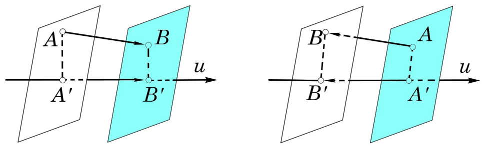


Figure 1.9: Projection of a vector on an axis.

The projection of a vector on an axis is a scalar called the **algebraic value of the projection** or shortly **projection**. Thus, the projection of a vector on an axis is a scalar. The algebraic value of the projection of the vector $\overrightarrow{AB}$ is denoted by $\overline{A'B'}$, and defined by:

$$\overline{A'B'} = \begin{cases} +\left|\overrightarrow{A'B'}\right|, & \text{if the vector } \overrightarrow{A'B'} \text{ has the same direction as the axis } u, \\ -\left|\overrightarrow{A'B'}\right|, & \text{if the vector } \overrightarrow{A'B'} \text{ and the axis } u \text{ have opposite directions.} \end{cases}$$

If the angle between the vector $\overrightarrow{AB}$ and the vector $\mathbf{u}$ of the axis u is denoted by α, then

$$\overline{A'B'} = \text{proj}_{\mathbf{u}}\overrightarrow{AB} = |\overrightarrow{AB}|\cos\alpha.$$

Note that the following **proposition** holds: the projection (algebraic value of the projection) of a sum of vectors on an arbitrary axis, is equal to the sum of the projections of these vectors, parts of the sum, on that axis.

Projection of a point and a vector on a plane

In order to project a point (A) on a plane (S), it is necessary to first select a straight line (p) with respect to which the point we will be projected. The intersection (A') of the plane (S) and the line (p_1), $(p \parallel p_1)$, to which point (A) belongs, is called the **projection of point** A **on a plane** (S) **in the direction of straight line** (p) (Fig. 1.10). If the line (p) is normal to the plane (S), then the corresponding projection is called **normal** (orthogonal).

The projection of a vector on a plane is obtained by projecting its start and end points (Fig. 1.10).

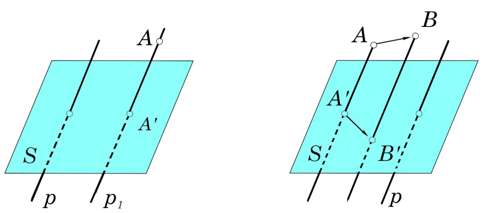


Figure 1.10: Projection of a point and a vector on a plane.

Thus, the projection of a vector on a plane is a vector.

1.4.4 Scalar (dot or internal) product of two vectors

Definition

The **scalar product** of two vectors **a** and **b**, symbolically denoted by $\mathbf{a}\cdot\mathbf{b}$ (which is read as "a dot b") or (**a b**), is a real number determined by: $|\mathbf{a}|\cdot|\mathbf{b}|\cdot\cos(\mathbf{a},\mathbf{b})$, i.e.

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cdot\cos\gamma, \tag{1.12}$$

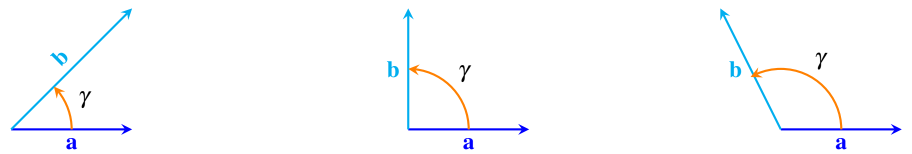
where γ is the angle between vectors **a** and **b**.

It follows from the very definition that the scalar product is equal to the **projection** of the vector **a** on the direction of the vector **b**, multiplied by the magnitude (length) of the vector **b**, i.e. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{b}|\cdot \mathrm{proj}_{\mathbf{b}}\mathbf{a}$. By analogy, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot \mathrm{proj}_{\mathbf{a}}\mathbf{b}$, given the commutativity of the scalar product and the parity of the $\cos\gamma$ function.

In mechanics (physics) the scalar product has the following physical meaning. If the force acting on some point M is denoted by **S**, and the elementary displacement of that point by d**r**, then the variable dA, defined by the relation

$$\mathrm{d}A = \mathbf{S}\cdot\mathrm{d}\mathbf{r}$$

represents the elementary work of the force **S** on the displacement d**r**.



(a) $\mathbf{a}\cdot\mathbf{b} > 0$ (b) $\mathbf{a}\cdot\mathbf{b} = 0$ (c) $\mathbf{a}\cdot\mathbf{b} < 0$

Figure 1.11: The sign of the scalar product - angle between the vectors (a) sharp (b) right (c) obtuse.

The sign of the scalar product depends on the angle between the vectors. Thus, the product is positive if the angle between vectors is sharp or zero, or if the vectors are orthogonal (right angle), and negative if the angle is obtuse (between $\pi/2 < \gamma < \pi$) (Fig. 1.11).

Starting from this definition the magnitude of a vector and the condition under which two vectors are orthogonal can be determined.

Namely, in the special case, when $\mathbf{a} = \mathbf{b}$, it follows that $\gamma = 0$ and, according to (1.12),

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|\cdot|\mathbf{a}|\cdot\cos(\mathbf{a},\mathbf{a}) = |\mathbf{a}|\cdot|\mathbf{a}| = |\mathbf{a}|^2 \Rightarrow \quad |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}. \tag{1.13}$$

Thus, it follows directly from the definition of the scalar product that the square of the vector magnitude is equal to the scalar product of the vector with itself.

It also follows from the definition of a scalar product for the angle γ between two vectors

$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|} \quad \Rightarrow \quad \gamma = \arccos\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}, \tag{1.14}$$

and thus for $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$ two vectors are orthogonal iff[9] $\mathbf{a}\cdot\mathbf{b}$=0.

From the previous definitions and properties of real numbers, the following properties, which are also called *metric properties of a linear vector space*, follow:

[9] iff is short for "if and only if" (necessary and sufficient condition).

- the scalar product of an arbitrary vector with itself is non-negative

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 > 0, \text{ and} \tag{VII}$$
$$\mathbf{a}\cdot\mathbf{a} = 0, \quad \text{if} \quad \mathbf{a} = \mathbf{0},$$

(positively – definite)

- the scalar product is commutative

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \tag{VIII}$$

(symmetry)

- the scalar product is distributive with respect to addition

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}, \tag{IX}$$

- the scalar product is associative with respect to multiplication by a scalar

$$\alpha(\mathbf{a}\cdot\mathbf{b}) = (\alpha\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(\alpha\mathbf{b}), \quad \text{where } \alpha \text{ is a real number.} \tag{X}$$

Some other properties that follow from the definition of a scalar product are:

$$|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}|\cdot|\mathbf{b}|, \tag{1.15}$$

(Schwarz inequality) [10]

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}|, \tag{1.16}$$

(triangle inequality)

$$|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2 = 2(|\mathbf{a}|^2+|\mathbf{b}|^2). \tag{1.17}$$

(parallelogram equality)

A real affine space $\mathbb{V}$ or real vector space in which the scalar product of a vector with properties (VII)–(X) is defined, is called the **real Euclidean**[11] **space**.

The concept of Euclidean space defined in this way is used to define a more general concept of Euclidean space.

A set $\mathbb{E}$, with elements of an arbitrary nature, for which the following is axiomatically defined:

1) an addition operation with properties (I)–(III),
2) an multiplication operation of elements of set $\mathbb{E}$ by elements of a field $\mathbb{R}$, with properties (IV)–(VI) and
3) a multiplication operation with properties (VII)–(X),

is called **Euclidean space over the field** $\mathbb{R}$.

Let us now define an orthonormal set of vectors.

[10] Hermann Amandus Schwarz (1843-1921), German mathematician, known for his work in complex analysis (conformal mapping), differential geometry and calculus of variations.

[11] $E\upsilon\kappa\lambda\varepsilon\iota\delta\eta\varsigma$, born about 330 BC, and died about 275 BC. One of the greatest Greek mathematicians of the ancient era. He was one of the founders and central figure of the mathematics school in Alexandria. He has written several works on geometry, optics and astronomy. His most important work is *Elements* ($\Sigma\tau o\iota\chi\varepsilon\tilde{\iota}\alpha$).

Definition

It is said that a set of three vectors (in 3-D Euclidean space) $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is an orthogonal normalized set or shortly **orthonormal set**, if the following condition is satisfied:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad i, j = 1, 2, 3. \tag{1.18}$$

The previous definition also applies in the n-dimensional Euclidean space E_n, where the indices i and j, in relation (1.18), take the values $i, j = 1, 2, \ldots, n$.

The variable δ_{ij}, defined by the previous relation, is referred to in the literature as Kronecker's[12] delta symbol.

1.4.5 Vector (cross) product of two vectors

Definition

A **vector product** of two vectors **a** and **b** in E_3 is a vector **c** determined by the following conditions:

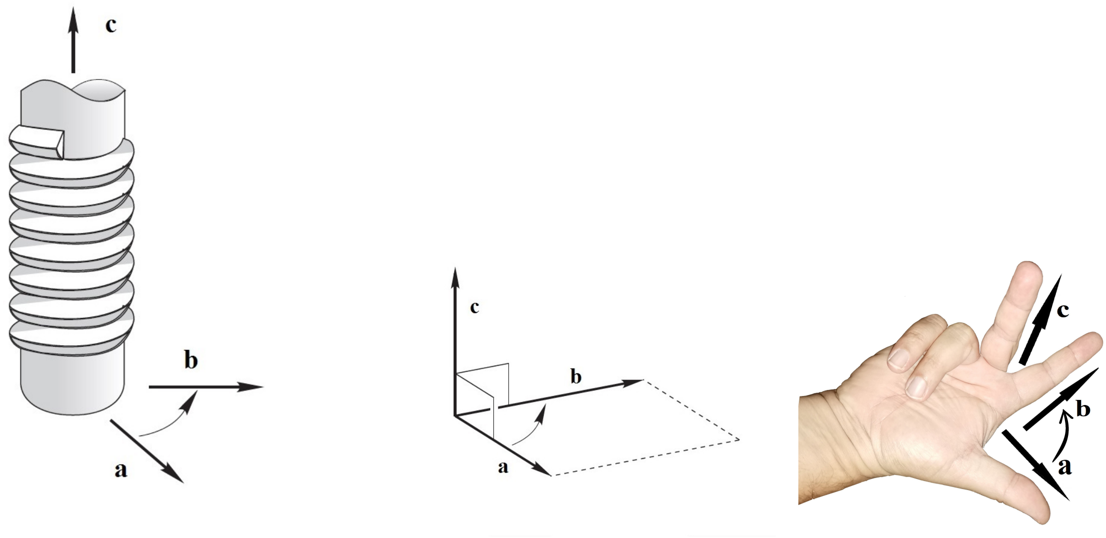
i) **c** is perpendicular to both **a** and **b**, and thus normal to the plane containing vectors **a** and **b**;

ii) **direction** of the vector **c** is given by the right-hand rule (or the right-screw rule). Namely, if we point the thumb of our right hand in the direction of vector **a**, and our index finger in the direction of vector **b**, and then rotate the vector **a** by a sharp angle (in the positive direction) to coincide with vector **b**, then the tip of the middle finger will indicate the direction of the vector product (see figures 1.12a, 1.12b and 1.12c);

iii) **magnitude** of the vector **c** is determined by the relation:

$$|\mathbf{c}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\alpha, \qquad \alpha = \angle(\mathbf{a}, \mathbf{b}). \tag{1.19}$$

[12]Leopold Kronecker (1823-1891), German mathematician, who gave a significant contribution to algebra, group theory and number theory.



(a) (b) (c)

Figure 1.12: Right-screw rule (a), and right-hand rule (c)

These conditions uniquely determine the vector **c**.

The vector product is symbolically denoted by:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}, \tag{1.20}$$

which is read as "a cross b".

In mechanics (physics) the vector product has the following physical meaning. Consider rotating a body around a fixed point. This rotation is due to the action of moment. The moment of force **S** for a point is defined by the following relation

$$\mathbf{M}_O^S = \mathbf{r} \times \mathbf{S},$$

where **r** is the position vector of the point of application of the force relative to the moment point O.

Note that the following holds for the vector product:

- it is *distributive* with respect to addition:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \tag{1.21}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \tag{1.22}$$

- it is *not commutative*, as (Fig. 1.13)

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad \textit{(anticommutativity)} \tag{1.23}$$

- it is *not associative*, as in general

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}. \tag{1.24}$$

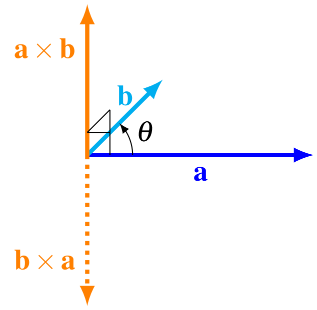


Figure 1.13: Anticommutativity of a vector product.

It follows from the definition of a vector product that the vector product of two vectors of the same direction is equal to zero, i.e.

$$\mathbf{a} \times \alpha\mathbf{a} = \mathbf{0}.$$

The previously given definition of a vector, with its corresponding operations, is a "geometric" definition. Namely, it follows from all the above that the vectors and the operations on them are independent of the choice of the coordinate system. In the text that follows, vectors will be observed “algebraically”, by defining their components with respect to a given coordinate system.

The product of three vectors

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}),$$

which is called **mixed product** is often used in practice. The product defined in this way is a scalar. It is obtained by initial vector multiplication of **b** and **c**, and then by scalar multiplication of the vector thus obtained and the vector **a**. The literature also uses the designation $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ for the product defined in this way.

For a mixed product, the property of **circular permutation** applies, namely

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}].$$

1.4.6 Reciprocal (conjugate) system of vectors

Definition

Two sets of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{a}_1', \ldots, \mathbf{a}_n'$ are said to represent a reciprocal or conjugate system if the scalar product of a vector form one set with a vector from another is given by the relation

$$\mathbf{a}_i \cdot \mathbf{a}_j' = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad i, j = 1, \ldots, n, \tag{1.25}$$

which can also be represented by the following table (for $n = 3$) of by the following figure, for $n = 2$ (Fig. 1.14).

$\bullet$	$\mathbf{a}'_1$	$\mathbf{a}'_2$	$\mathbf{a}'_3$
$\mathbf{a}_1$	1	0	0
$\mathbf{a}_2$	0	1	0
$\mathbf{a}_3$	0	0	1

Table 1.1: Reciprocal bases vectors.

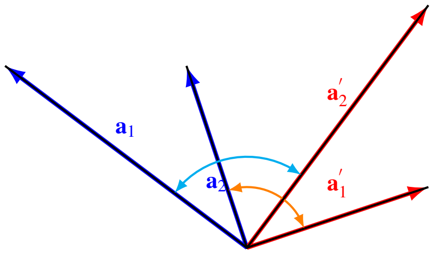


Figure 1.14: Reciprocal vectors in 2D.

1.4.7 Linear dependence of vectors. Dimension of a space

Let us now introduce the term linear dependence of a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$.

Definition

Vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ are **linearly dependent** if there exist numbers $\alpha_1, \cdots, \alpha_n$, at least one of which is different from zero, such that the following relation holds

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_n \mathbf{a}_n = 0. \tag{1.26}$$

Conversely, the vectors are **linearly independent**, if the relation (1.26) is true only when

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0, \tag{1.27}$$

Definition

A vector space is n–**dimensional** if it contains n linearly independent vectors, while each system of $n+1$ vectors is linearly dependent.

Let us illustrate this with a few examples.
Consider two vectors **a** and **b** with the same or opposite directions (Fig. 1.15)

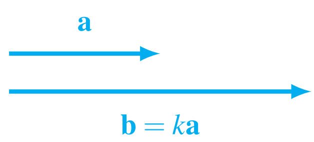


Figure 1.15: Collinear vectors.

Then a (real) number $k \neq 0$ exists such that:

$$\mathbf{b} = k\mathbf{a}, \tag{1.28}$$

and vectors **a** and **b** are called **collinear** vectors.

Assuming $k = -\frac{\alpha}{\beta}$, the relation (1.28) can be represented as:

$$\alpha \mathbf{a} + \beta \mathbf{b} = 0. \tag{1.29}$$

It can be concluded that the two collinear (or parallel) vectors are linearly dependent, since α and β are different from zero. Thus, it can be said that all vectors $k\mathbf{a}$, for arbitrary and real k and $\mathbf{a} \neq 0$, form a one-dimensional (1–D) real linear vector space. Such terminology is used due to the fact that to each point on the axis a position vector[13] can be assigned and conversely, to each vector from this set a point on the axis corresponds (one-to-one correspondence).

Consider now two non-collinear vectors **a** and **b**. Let us represent them by oriented segments with a common beginning O (Fig. 1.16).
An arbitrary vector **c**, lying in the plane of vectors **a** and **b**, can be represented in the form

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}. \tag{1.30}$$

This relation follows from the vector addition rules and from the definition of multiplication of a vector by a scalar. From relation (1.30), similar as in the case of (1.28) and (1.29), and assuming:

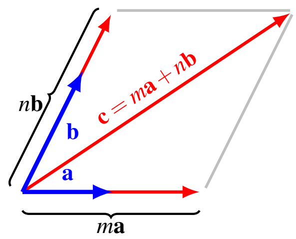


Figure 1.16: Non-collinear vectors.

$$m = -\frac{\alpha}{\gamma}, \quad n = -\frac{\beta}{\gamma}, \tag{1.31}$$

we obtain

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0, \tag{1.32}$$

which is the condition for linear dependence of a set of three vectors, because not all constants are zero. In this way, each point in the plane can be determined by a position vector **c**, i.e. by a combination of the vectors $m\mathbf{a} + n\mathbf{b}$, where **a** and **b** are two linearly independent vectors, and m and n are the corresponding real numbers. Therefore, it can be said that the combination $m\mathbf{a} + n\mathbf{b}$ defines a two-dimensional (2–D) real linear vector space. It can also be noted that in a 2–D linear vector space a set of three vectors is always linearly dependent.

Consider now three non-coplanar[14] vectors **a**, **b** i **c**, starting from a common origin O (sl. 1.17).

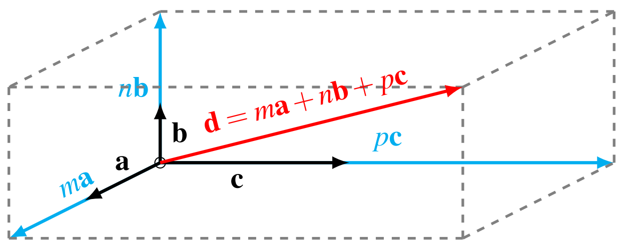


Figure 1.17: Sum of vectors in $3-D$.

As in the previous cases, any subsequent vector **d** can be represented by the relation

$$\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}, \tag{1.33}$$

[13]The position vector of a point A is the vector $\mathbf{r}_A = \overrightarrow{OA}$, that starts in the origin O and ends in point A.
[14]Vectors are coplanar if they are all parallel to one plane.

whence it follows that between four vectors **a**, **b**, **c** and **d** there is always a nontrivial relation of the form

$$\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0. \tag{1.34}$$

Thus, the relation (1.33), for each set of real numbers m, n, and p, determines a three-dimensional linear vector space. One can imagine that the end point of vector **d** is "overwriting" all points of the 3–D space, when the parameters m, n, and p are taking all possible values from the set of real numbers. This means that in a 3–D linear vector space, each set of four vectors is linearly dependent. We will use this relation between the number of linearly independent vectors and the dimension of a space to introduce the concept of dimensionality of a three-dimensional linear vector space, noting that the concept can easily be generalized to an n–dimensional vector space.

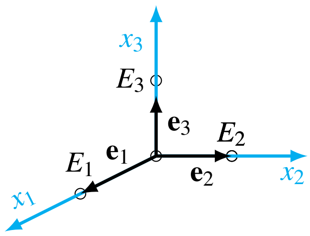
The vectors **a**, **b** and **c**, in (1.32) are called **base vectors**, and the elements of the sum $m\mathbf{a}$, $n\mathbf{b}$ and $p\mathbf{c}$ **components of the vector d**. Numbers m, n and p will shortly be called **coordinates**[15] with respect to base vectors **a**, **b** and **c**.

Once a set of base vectors is determined, then each vector is uniquely determined by a triple (in 3–D) of coordinates.

A set of three mutually orthogonal vectors in 3–D space is linearly independent[16]. If orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are chosen as the base vectors, then each (subsequent) vector, e.g. **x**, can be represented by the relation

$$\mathbf{x}=x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3. \tag{1.35}$$

A point in 3–D space is a geometric object (does not depend on the coordinate system). If we introduce a coordinate system, we can uniquely determine each point by an ordered triple of numbers (x_1,x_2,x_3), whose elements are called **vector coordinates** (hereafter shortly **coordinates**) of **x**. It is said that the vectors $\mathbf{e}_i$, i=1,2,3, form a base or coordinate system (Fig. 1.18). The vectors ($\mathbf{e}_i$) are called (as already mentioned) base vectors.



The end points E_i of the base vectors $\mathbf{e}_i$ $(i=1,2,3)$ have the following coordinates:

$$\begin{aligned} E_1 &: \quad (1,0,0),\\ E_2 &: \quad (0,1,0),\\ E_3 &: \quad (0,0,1). \end{aligned} \tag{1.36}$$

Figure 1.18: Base vectors and their coordinates.

Namely, vectors have previously been defined geometrically, using the oriented segment. By introducing the coordinate system, the vector can be described algebraically. It has already been

[15] Note that in spaces where a scalar product is not defined, such as an affine space, there is no point in considering concepts that are defined using this product, such as magnitude or angle between two vectors. It is common in the literature that these variables, which we have called coordinates, are also called affine coordinates, thus emphasizing the nature of this (affine) space.

[16] Observe a set of three mutually orthogonal vectors, for which $\mathbf{a}_i\cdot\mathbf{a}_j=A_{ij}\delta_{ij}$, where $A_{ij}=|\mathbf{a}_i|\cdot|\mathbf{a}_j|$. Let us assume that the linear combination of these vectors $\sum_{i=1}^{3}\lambda_i\mathbf{a}_i=0$. By a scalar multiplication of the last relation with $\mathbf{a}_j$ $(j=1,2,3)$, and taking into account the condition of orthogonality, we obtain

$$\sum_{i=1}^{3}\lambda_i\mathbf{a}_i\cdot\mathbf{a}_j=\sum_{i=1}^{3}\lambda_i(\mathbf{a}_i\cdot\mathbf{a}_j)=\sum_{i=1}^{3}\lambda_iA_{ij}\delta_{ij}=\lambda_jA_{jj}=0 \quad\Rightarrow\quad \lambda_j=0,$$

which is the condition for linear independence of the observed vectors.

said that a coordinate system with mutually perpendicular axes is called the Cartesian coordinate system. It is common to denote the axes of the Cartesian coordinate system by x, y and z, instead of x_1, x_2 and x_3, respectively, and the corresponding base vectors by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, instead of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, respectively. Note that both the left and the right coordinate systems are used, although the right one is more common (Fig. 1.19).

(a) Right orientation. (b) Left orientation.

Figure 1.19: Orientation of coordinate systems.

Consider now an arbitrary vector $\mathbf{a}$, represented by the oriented segment $\overrightarrow{AB}$, where A is the start, and B the end of segment AB (Fig. 1.20).

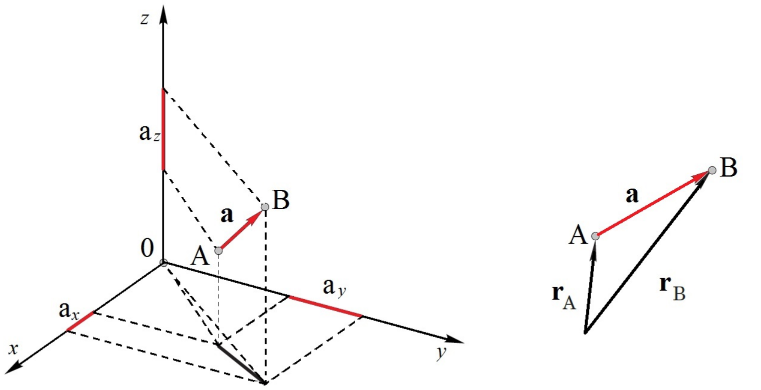


Figure 1.20: Projection of vectors expressed in terms of coordinates of their start and end points.

If two points A (x_A, y_A, z_A) and B (x_B, y_B, z_B) are given by their coordinates, and the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are the position vectors of these points, then:

$$\overrightarrow{AB} = \mathbf{a} = \mathbf{r}_B - \mathbf{r}_A \quad \Rightarrow$$
$$a_x = x_B - x_A,\ a_y = y_B - y_A \ \text{i}\ a_z = z_B - z_A \tag{1.37}$$

where a_x, a_y and a_z are measures of the vector $\mathbf{a}$ with respect to the coordinate system, which is shortly denoted, for simplicity, by

$$\mathbf{a} = [a_x,\, a_y,\, a_z] \tag{1.38}$$

instead by

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Let us now express the previously defined concepts through the corresponding measures.

The magnitude of a vector $\mathbf{a}$ is, by its definition, the distance between points A and B, which, according to (1.3), can be represented in the Euclidean space by the relation

$$|\mathbf{a}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.39}$$

If the origin is chosen for the beginning of the vector, then the coordinates of the end point of the vector are equal to the measures of the vector. It has already been said that a vector constructed in this way is called **position vector** and is usually denoted by **r**.

It can also be observed from (1.37) that the measures a_x, a_y, a_z of the vector **a** do not depend on the choice of the start point of **a**, because if the vector **a** is moved along the direction AB, then the coordinates of the points A and B change for the same value, and their difference remains the same. Thus, if a fixed Cartesian coordinate system is given, then each vector is uniquely determined by an ordered triple of numbers (coordinates). The zero vector **0** can defined accordingly as a vector whose coordinates are $[0,0,0]$.

It is said that two vectors $\mathbf{a}=[a_x, a_y, a_z]$ and $\mathbf{b}=[b_x, b_y, b_z]$ are equal iff their respective coordinates are equal. Namely, the vector equation:

$$\mathbf{a} = \mathbf{b} \tag{1.40}$$

is equivalent to three scalar equations:

$$a_x = b_x, \quad a_y = b_y, \quad a_z = b_z. \tag{1.41}$$

The following operations can now be defined using the coordinates:
Sum of vectors (Fig. 1.21):

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = [a_x + b_x, a_y + b_y, a_z + b_z] = [c_x, c_y, c_z] \tag{1.42}$$

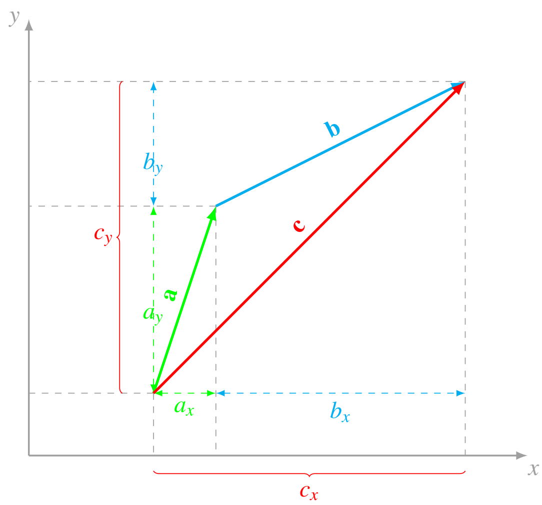


Figure 1.21: Sum of vectors expressed in terms of their projections (components).

Multiplication of a vector by a scalar

$$\alpha\mathbf{a} = [\alpha a_x, \alpha a_y, \alpha a_z] \tag{1.43}$$

Scalar product

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.44}$$

given that

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{1.45}$$

which follows from the assumption that the base vectors are orthonormal.

Vector product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{1.46}$$

The vector product can be represented by the above symbolic determinant, as shown in exercise 4, on p. 56. Namely, the vector product $\mathbf{a}\times\mathbf{b}$ represented in this way is equal to the expression obtained by developing this determinant by the first row.

Mixed product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.47}$$

Note that a mixed product is zero if these three vectors are coplanar, i.e. linearly dependent. In particular, if $\mathbf{a}$ is collinear, with say, $\mathbf{b}$ then $\mathbf{a} = \lambda\mathbf{b}$, and by substitution in (1.47) we obtain

$$\mathbf{a}\cdot(\lambda\mathbf{a}\times\mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ \lambda a_x & \lambda a_y & \lambda a_z \\ c_x & c_y & c_z \end{vmatrix} = \lambda \begin{vmatrix} a_x & a_y & a_z \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix} = 0.$$

We have used here the following properties of a determinant
- a determinant is multiplied by a number by multiplying elements of one row or column by that number and
- a determinant is equal to zero if any two rows or columns are equal.

We would obtain the result in a similar way in the case of collinearity of vectors **a** and **c**.

1.5 Algebraic model of linear vector space

We are familiar with the use of "ordinary" vectors in three-dimensional space to represent physical quantities, such as: position vector, velocity, acceleration, force, etc. We will now define an abstract **linear vector space** using the well-known properties of such vectors.

Definition

Let $\mathbb{X}$ be a nonempty set whose elements $\mathbf{x},\mathbf{y},\mathbf{z},\ldots$, will be called vectors, and T a set of all real (complex) numbers, whose elements $\alpha,\beta,\ldots$, will be called scalars. The pair $(\mathbb{X},T)$ forms a **linear vector space** or shortly vector space (real or complex, depending on the set of scalars T), if it has the following algebraic structure

i) to each ordered pair of vectors $(\mathbf{x},\mathbf{y})$ from $\mathbb{X}$ corresponds a third vector $\mathbb{X}$, which will be called their sum, and denoted by $\mathbf{x}+\mathbf{y}$. The operation that assigns vector $\mathbf{x}+\mathbf{y}$ to the ordered pair $(\mathbf{x},\mathbf{y})$ will be called **addition of vectors**.

ii) To each vector $\mathbf{x}\in\mathbb{X}$ and each scalar $\alpha\in T$ corresponds a vector from $\mathbb{X}$, which will be called product of the vector $\mathbf{x}$ by the scalar α, and denoted by $\alpha\mathbf{x}$. The operation that associates vector $\mathbf{x}$ from $\mathbb{X}$ and scalar α to the vector $\alpha\mathbf{x}$ is called **multiplication of a vector by a scalar**.

The operations addition of vectors and multiplication of a vector by a scalar have the following properties

1° addition is commutative

$$\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}, \tag{1.48}$$

2° addition is associative

$$(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z}), \tag{1.49}$$

3° in $\mathbb{X}$ there exists a zero vector $\mathbf{0}$, such that

$$\mathbf{x}+\mathbf{0}=\mathbf{x}, \quad \text{for each } \mathbf{x}\in\mathbb{X}, \tag{1.50}$$

4° to each vector $\mathbf{x}\in\mathbb{X}$ corresponds an opposite vector in $\mathbb{X}$, denoted by $-\mathbf{x}$, such that

$$\mathbf{x}+(-\mathbf{x})=\mathbf{0}. \tag{1.51}$$

Multiplication is distributive

5° with respect to the addition of vectors

$$\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}, \tag{1.52}$$

6° and with respect to the addition of scalars

$$(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}, \tag{1.53}$$

7° multiplication by a scalar is associative

$$\alpha(\beta\mathbf{x})=(\alpha\beta)\mathbf{x}, \tag{1.54}$$

8° multiplication of a vector by the scalar 1 leaves the vector unchanged i.e.

$$1\mathbf{x}=\mathbf{x}, \quad \text{where} \quad 1\in T. \tag{1.55}$$

Definition

A vector space $(\mathbb{X},T)$ is **normed** if there exists a nonnegative function $\|\mathbf{x}\|$, defined for each $\mathbf{x}\in\mathbb{X}$, with the following properties

$$\|\mathbf{0}\|=0 \quad \text{and} \quad \|\mathbf{x}\|>0, \quad \text{for} \quad \mathbf{x}\neq\mathbf{0}, \tag{1.56}$$

$$\|\lambda\mathbf{x}\|=|\lambda|\cdot\|\mathbf{x}\|, \quad \text{for each} \quad \lambda\in T, \tag{1.57}$$

$$\|\mathbf{x}+\mathbf{y}\|\leq\|\mathbf{x}\|+\|\mathbf{y}\| \quad \text{(rule of the triangle).} \tag{1.58}$$

This function is called the **norm** of $\mathbf{x}$.

Definition

Let $\mathbb{X}$ be a set whose elements are denoted by $\mathbf{x},\mathbf{y}\ldots$. If to each ordered pair $(\mathbf{x},\mathbf{y})$ from $\mathbb{X}$ a real number $d(\mathbf{x},\mathbf{y})$ is assigned, with the following properties

$$0\leq d(\mathbf{x},\mathbf{y})<+\infty, \tag{1.59}$$

$$d(\mathbf{x},\mathbf{y})=0\Leftrightarrow\mathbf{x}=\mathbf{y}, \tag{1.60}$$

$$d(\mathbf{x},\mathbf{y})=d(\mathbf{y},\mathbf{x}), \tag{1.61}$$

$$d(\mathbf{x},\mathbf{y})\leq d(\mathbf{x},\mathbf{z})+d(\mathbf{z},\mathbf{y}), \tag{1.62}$$

then it is said that the set $\mathbb{X}$ is equipped with the metric d. A set $\mathbb{X}$ equipped with the metric d is called a **metric space**. Its elements are called points, and $d(\mathbf{x},\mathbf{y})$ is called the **distance** between points $\mathbf{x}$ and $\mathbf{y}$. A metric space is, thus, a pair $(\mathbb{X},d)$.

In a normed vector space the metric is introduced by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|. \tag{1.63}$$

Let us denote by $\mathbb{R}^n$ a set whose points are ordered n–tuples of real (complex) numbers and introduce a metric in this space by

$$d(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}. \tag{1.64}$$

The space $\mathbb{R}^n$ is called Euclidian (real or complex) metric space, and d defined in this way satisfies the conditions (1.59)–(1.62).

Let us now introduce the concept of linear operator. Consider a linear vector function, of a vector variable, which assigns to each vector $\mathbf{x}$ another vector $A(\mathbf{x})$, and for which the following is true

$$A(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha A(\mathbf{x})+\beta A(\mathbf{y}), \tag{1.65}$$

where α and β are scalars, and $\mathbf{x}$ and $\mathbf{y}$ vectors. The function defined in this way is called **linear operator**.

A linear operator is fully determined if vectors $A(\mathbf{e}_i)$ are given, where vectors $\mathbf{e}_i$ $(i = 1,2,\ldots,n)$ form a set of base vectors. Vectors $A(\mathbf{e}_i)$ can be expressed using the vectors $\mathbf{e}_i$

$$A(\mathbf{e}_i) = \sum_{j=1}^{n} A_{ji}\mathbf{e}_j, \tag{1.66}$$

where A_{ji} is the j–th component of the vector $A(\mathbf{e}_i)$. Let us now consider an arbitrary vector $\mathbf{x}$, and denote $A(\mathbf{x})$ by $\mathbf{y}$, i.e. $A(\mathbf{x}) = \mathbf{y}$. Then the following relations can be established using (1.65) and (1.66). Let us first express $\mathbf{y}$ using base vectors

$$\mathbf{y} = \sum_{j=1}^{n} y_j\mathbf{e}_j.$$

Given that

$$\begin{aligned}\mathbf{y} = A(\mathbf{x}) = A\left(\sum_{i=1}^{n} x_i\mathbf{e}_i\right) &= \\ = \sum_{i=1}^{n} x_i A(\mathbf{e}_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{ji}\mathbf{e}_j &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} A_{ji}x_i\right)\mathbf{e}_j,\end{aligned} \tag{1.67}$$

it follows that vector coordinates $\mathbf{x}$ and $\mathbf{y}$ are bound by the relation

$$y_j = \sum_{i=1}^{n} A_{ji}x_i. \tag{1.68}$$

This can also be expressed in the following way. Let us assume that we have established the relation between vectors $\mathbf{x}$ and $\mathbf{y}$ using a linear operator A applied to $\mathbf{x}$. This can be denoted symbolically by[17]

$$\mathbf{y} = A\mathbf{x}. \tag{1.69}$$

[17]Note that in the case of linear operator A, the symbol $A\mathbf{x}$ is used on an equal basis.

The numbers A_{ji} are called components of linear operator A (or vector function A) in the coordinat system $\mathbf{e}_i$. Specifically, it follows from (1.66) that A_{ji} is the j–th component of the vector $A\mathbf{e}_i$.

Similar as vectors, **linear operators** often have their physical meaning, which is independent of a specific coordinate system, and can be described without a coordinate system. This can be expressed symbolically as follows.

The **addition** and **multiplication** of linear operators and **multiplication of a linear operator by a scalar** can be defined by the following relations

$$(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} \tag{1.70}$$
$$(AB)\mathbf{x} = A(B\mathbf{x}) \tag{1.71}$$
$$(\lambda A)\mathbf{x} = \lambda(A\mathbf{x}), \tag{1.72}$$

for each $\mathbf{x} \in \mathbb{X}$, where $\mathbb{X}$ represents a vector space.

In the general case $AB \neq BA$. If the products are equal, then it is said that the "multiplication" is commutative.

Let is define the zero (0) operator and the identity operator (I) by the following relations

$$0\mathbf{x} = 0 \quad \text{and} \quad I\mathbf{x} = \mathbf{x}, \tag{1.73}$$

for each $\mathbf{x}$ from the observed space.

Two operators are equal if the following is true

$$A\mathbf{x} = B\mathbf{x} \tag{1.74}$$

for each vector $\mathbf{x} \in \mathbb{X}$.

Finally, if there exists an operator A^{-1} with the following property

$$AA^{-1} = A^{-1}A = I, \tag{1.75}$$

than this operator $\left(A^{-1}\right)$ is called **inverse** operator for operator A. Operators that have inverse operators are called **nonsingular**.

1.6 Gram-Schmidt orthogonalization procedure

Theorem 1

If the non-zero vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^k$ are mutually orthogonal, then the set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly dependent.

Proof

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{0} \,|\cdot \mathbf{u}_i$$
$$c_i\left(\mathbf{u}_i \cdot \mathbf{u}_i\right) = \mathbf{0} \quad \Rightarrow \quad c_i = 0,$$

because $\mathbf{u}_i$ is not a zero-vector, i.e. $\mathbf{u}_i \cdot \mathbf{u}_i > 0$, the other members are equal to zero due to the orthogonality among vectors.

Definition

Orthogonal basis for the vector space is a basis in which all vectors $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ are mutually orthogonal.

Theorem 2

Let $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ be an orthogonal basis in $\mathbb{R}^k$. Each vector in this space can be represented as $\mathbf{u} = c_1\,\mathbf{u}_1 + \cdots + c_k\,\mathbf{u}_k$, where the set $\{c_1,\dots,c_k\}$ is a representation of the vector $\mathbf{u}$ in space $\mathbb{R}^k$ with the basis $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$. Then, for each $i \in \{1,\dots,k\}$, $c_i = \mathbf{u}\cdot\mathbf{u}_i/(\mathbf{u}_i\cdot\mathbf{u}_i)$ is the Fourier coefficient of vector $\mathbf{u}$ with respect to vector $\mathbf{u}_i$.

Proof

$$\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k \,|\cdot\mathbf{u}_i \quad\Rightarrow\quad \mathbf{u}\cdot\mathbf{u}_i = c_i\,(\mathbf{u}_i\cdot\mathbf{u}_i) \quad\Rightarrow$$
$$c_i = \frac{\mathbf{u}\cdot\mathbf{u}_i}{(\mathbf{u}_i\cdot\mathbf{u}_i)}.$$

Theorem 3 (Gram-Schmidt orthogonalization)

If $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}$, for $k \geq 1$, is the basis pf space $\mathbb{R}^k$, then the vectors

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{u}_1,\\
\mathbf{p}_2 &= \mathbf{u}_2 - \mathrm{proj}(\mathbf{u}_2,\mathbf{p}_1),\\
\vdots &= \vdots,\\
\mathbf{p}_k &= \mathbf{u}_k - \mathrm{proj}(\mathbf{u}_k,\mathbf{p}_1) - \cdots - \mathrm{proj}(\mathbf{u}_k,\mathbf{p}_{k-1})
\end{aligned}$$

form the orthogonal basis of that space.

Proof

The proof consists of two parts:
- part 1 prove that vectors $\mathbf{p}_k$ are not zero-vectors,
- part 2 prove that the vectors $\mathbf{p}_1,\dots,\mathbf{p}_k$ are mutually orthogonal, and thus, according to Theorem 1, it follows that $\{\mathbf{p}_1,\dots,\mathbf{p}_k\}$ form an orthogonal basis of space $\mathbb{R}^k$.

Part 1 will be proved using the principle of mathematical induction:

- step one: for $k = 1$

Given that $\mathbf{p}_1 = \mathbf{u}_1$, the Theorem is true.

- step two: it is true for $k-1 \geq 1$, i.e.

vectors $\mathbf{p}_1,\dots,\mathbf{p}_{k-1}$ are mutually orthogonal, non-zero vectors in a space whose basis is $\{\mathbf{u}_1,\dots,\mathbf{u}_{k-1}\}$.

- step three: prove that it is true for k also.

Let us assume the opposite, i.e. that $\mathbf{p}_k$ is a zero vector ($\mathbf{p}_k = \mathbf{0}$), then the vector $\mathbf{u}_k$ can be represented in terms of vectors $\mathbf{p}_1, \dots, \mathbf{p}_{k-1}$

$$\mathbf{u}_k = \text{proj}\,(\mathbf{u_k}, \mathbf{p}_1) + \cdots + \text{proj}\,(\mathbf{u}_k, \mathbf{p}_{k-1}),$$

and it follows that the vector $\mathbf{u}_k$ belongs to a space whose basis is $\{\mathbf{p}_1, \dots, \mathbf{p}_{k-1}\}$, and which is a sub-space of a space whose basis is $\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$, which is contradictory to the assumption that the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent, which proves that the vector $\mathbf{p}_k$ is not a zero-vector.

Given that vectors $\mathbf{p}_1, \dots, \mathbf{p}_{k-1}$ belong to space w̌hose basis is $\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ it follows that

$$\mathbf{p}_k = \mathbf{u}_k - c_1\,\mathbf{p}_1 - \cdots - c_{k-1}\,\mathbf{p}_{k-1}. \tag{1.76}$$

From here, it follows that $\mathbf{p}_k$ belong to a space whose basis is $\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$.

Part two. It needs further to be proved that these vectors are mutually orthogonal, i.e. that $\mathbf{p}_k \perp \mathbf{p}_n$, for each $n = 1, \dots, k-1$.

Given that $\mathbf{p}_k$ is not a zero-vector it can be represented in the form (see (1.76))

$$\mathbf{p}_k = \mathbf{u}_k - c_1\,\mathbf{p}_1 - \cdots - c_{k-1}\,\mathbf{p}_{k-1}$$

and by scalar multiplication of the equation by vector $\mathbf{p}_n$ the following expression is obtained

$$\mathbf{p}_k \cdot \mathbf{p}_n = \mathbf{u}_k \cdot \mathbf{p}_n - c_n\,(\mathbf{p}_n \cdot \mathbf{p}_n).$$

Vector $\mathbf{u}_k$ can be represented in the form

$$\mathbf{u}_k = c_1\,\mathbf{p}_1 + \cdots + c_k\,\mathbf{p}_k,$$

and, on basis of Theorem 2, it follows that

$$\mathbf{u}_k \cdot \mathbf{p}_n = c_n\,\mathbf{p}_n \cdot \mathbf{p}_n \quad \Rightarrow \quad c_n = \frac{\mathbf{u}_k \cdot \mathbf{p}_n}{\mathbf{p}_n \cdot \mathbf{p}_n}.$$

Substituting c_n from this equation into the previous one, we obtain

$$\begin{aligned}\mathbf{p}_k \cdot \mathbf{p}_n &= \mathbf{u}_k \cdot \mathbf{p}_n - \mathbf{u}_n \cdot \mathbf{p}_k\,\frac{(\mathbf{p}_n \cdot \mathbf{p}_n)}{(\mathbf{p}_n \cdot \mathbf{p}_n)} = \\ &= \mathbf{u}_k \cdot \mathbf{p}_n - c_n\,(\mathbf{p}_n \cdot \mathbf{p}_n) = \\ &= \mathbf{u}_k \cdot \mathbf{p}_n - \frac{\mathbf{u}_k \cdot \mathbf{p}_n}{\mathbf{p}_n \cdot \mathbf{p}_n}\,(\mathbf{p}_n \cdot \mathbf{p}_n) = 0.\end{aligned}$$

Thus, the scalar product of these vectors is $(\mathbf{p}_k \cdot \mathbf{p}_n) = 0$, which means that these two vectors are mutually orthogonal ($\mathbf{p}_k \perp \mathbf{p}_n$).

Definition

The **Basis** of space $\mathbb{R}^k$ is **orthonormalized**, when this basis is composed of a orthonormalized set of vectors.

Theorem 4

If the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormalized basis, then the norm of the vector $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k$, is given by the expression $\|\mathbf{v}\| = \sqrt{c_1^2 + \cdots + c_k^2}$.

Proof

$$\|\mathbf{v}\| = \sqrt{(c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k) \cdot (c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k)} = \sqrt{c_1^2 + \cdots + c_k^2}$$

2. Vector analysis

2.1 Vector analysis

2.1.1 Vector function

Let T be a set of real (complex) numbers (scalars), and $\mathbb{V}$ a vector set.

Definition

If to each number $t \in T$, according to a certain rule, corresponds a specific value of a vector $\mathbf{v} \in \mathbb{V}$, then $\mathbf{v}$ is called a **vector function** of a scalar argument t and shortly denoted by

$$\mathbf{v} = \mathbf{v}(t). \tag{2.1}$$

The vector function can also be defined in another way:
a single-valued mapping of a set of real (complex) numbers T to a vector set $\mathbb{V}$, according to a certain rule

$$\mathbf{v} = \mathbf{v}(t). \tag{2.2}$$

is called a vector function $\mathbf{v}(t)$ of one scalar argument t.

The set of real (complex) numbers (scalars) T on which the function is defined is called the **domain** of the function $\mathbf{v}(t)$.

As, according to 1.38, $\mathbf{v}=[v_1, v_2, v_3]$ (in 3–D), the single-valued mapping of set T on set $\mathbb{V}$ consequently amounts to a mapping of the first set to the second set by means of three scalar functions, which represent the projections of the vector function $\mathbf{v}(t)$ on the coordinate axes

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)]. \tag{2.3}$$

On basis of the aforementioned, the analysis of vector functions of one scalar argument amount to the analysis of three (in 3–D) scalar functions–projections of vector $\mathbf{v}(t)$ on the axes of the corresponding coordinate system.

2.1.2 Hodograph of a vector function

Definition

Hodograph of a vector function $\mathbf{v}=\mathbf{v}(t)$ is the geometric location of the end points of the vector $\mathbf{v}$ for all possible values of t, where all these vectors start at a single fixed point, e.g. O, which is called the **pole of the hodograph**.

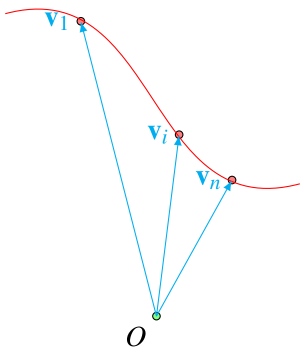


Figure 2.1: Hodograph of a vector function.

If the vector function represents a position vector (in 3–D), then the hodograph of this vector function represents a spatial curve (3–D). The vector equation of this curve is denoted by

$$\mathbf{r} = \mathbf{r}(t). \tag{2.4}$$

To this function three scalar (parametric) equations correspond, which represent the equation of the curve in space, in parametric form

$$x = x(t), \quad y = y(t), \quad z = z(t), \tag{2.5}$$

2.1.3 Limit processes. Continuity

Basic concepts in vector analysis, such as *convergence* and *continuity* can be introduced as follows.

Definition

It is said that a series of vectors $\mathbf{a}_n$, $n = 1, 2, \ldots,$ **converges** if there exists a vector $\mathbf{a}$ such that

$$\lim_{n\to\infty} |\mathbf{a}_n - \mathbf{a}| = 0. \tag{2.6}$$

The vector $\mathbf{a}$ is called the **limit vector** of the series $\mathbf{a}_n$ and denoted by

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}. \tag{2.7}$$

If a coordinate system is introduced, then it is said that the series of vectors $\mathbf{a}_n$ converges to $\mathbf{a}$ iff the three series (in 3-D) of vector components converge to the respective components of vector $\mathbf{a}$.

Definition

It is said that the vector function $\mathbf{v}(t)$ has vector ℓ as its **limit value** when the argument t tends to t_0, if for a given number $\varepsilon > 0$ a number $\delta = \delta(\varepsilon) > 0$ can be determined, such that for each t, for which

$$|t - t_0| < \delta, \tag{2.8}$$

the following relation holds

$$|\mathbf{v}(t) - \ell| < \varepsilon. \tag{2.9}$$

This can be symbolically denoted by

$$\lim_{t \to t_0} \mathbf{v}(t) = \ell \quad \text{or} \quad \mathbf{v}(t) \to \ell \quad \text{when} \quad t \to t_0. \tag{2.10}$$

Definition

A vector function $\mathbf{v}(t)$ is **continuous** in point $t = t_0$, if

1° it is defined in point $t = t_0$,

2° has a limit value when $t \to t_0$, and

3° if

$$\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0). \tag{2.11}$$

Let $\mathbf{v}(t)$ and $\mathbf{v}(t + \Delta t)$ be the values of the vector function $\mathbf{v}$ for argument values t and $t + \Delta t$ (Δt is the increment of the argument). The difference

$$\mathbf{v}(t + \Delta t) - \mathbf{v}(t) = \Delta \mathbf{v}(t) \tag{2.12}$$

is called the *geometric increment* of the function $\mathbf{v}(t)$. Let us represent this on the hodograph of the function $\mathbf{v}(t)$, Fig. 2.2. The geometric increment is represented by the vector $\overrightarrow{AB}$.

There is also another way of defining continuity.

Definition

A vector function $\mathbf{v}(t)$ is continuous for a given value of the argument t, if its geometric increment $\Delta \mathbf{v}$ tends to zero, when the increment of the argument tends to zero, i.e. when

$$\lim_{\Delta t \to 0} |\Delta \mathbf{v}(t)| = 0. \tag{2.13}$$

As $\mathbf{v}(t)$=$[v_1(t), v_2(t), v_3(t)]$, the necessary and sufficient condition for the continuity of the function $\mathbf{v}(t)$ in point t is that the projections $v_i(t)$ of this vector are continuous functions in this point, i.e., that:

$$\lim_{\Delta t \to 0} \Delta v_i(t) = 0 \quad \Rightarrow \quad \lim_{\Delta t \to 0} [v_i(t + \Delta t) - v_i(t)] = 0, \quad i = 1, 2, 3. \tag{2.14}$$

From here, the continuity of the function $\mathbf{v}(t)$ also follows, i.e.

$$|\Delta \mathbf{v}(t)| = \sqrt{(\Delta v_1(t))^2 + (\Delta v_2(t))^2 + (\Delta v_3(t))^2} \to 0. \tag{2.15}$$

2.1.4 Derivative of a vector function of one scalar variable

Observe now some (arbitrary) value of the scalar t and the corresponding value of the vector $\mathbf{v}(t)$. This vector is represented in Fig. 2.2 by the oriented segment $\overrightarrow{OA}$.

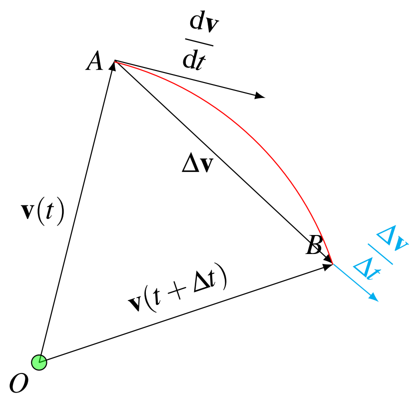


Figure 2.2: The increment of the vector function.

Let us now increase the value of the scalar t by the value of Δt. The vector that corresponds to value $t+\Delta t$ of the scalar is denoted by $\mathbf{v}(t+\Delta t)$. In Fig 2.2 this vector is represented by the oriented segment $\overrightarrow{OB}$. The change of the vector $\mathbf{v}(t)$, which corresponds to the increment of the scalar t by Δt, is given by the difference

$$\Delta\mathbf{v} = \mathbf{v}(t+\Delta t) - \mathbf{v}(t). \tag{2.16}$$

It can be observed in Fig. 2.2 that this difference, the geometric increment, is represented by the oriented segment $\overrightarrow{AB}(=\Delta\mathbf{v})$.

Consider now the vector $\frac{\Delta\mathbf{v}}{\Delta t}$, which represents the mean change in the value of $\mathbf{v}$ with respect to the parameter t. The vector defined in this way has the same direction as the vector $\Delta\mathbf{v}$ (Δt is a scalar) if $\Delta t > 0$, and opposite direction if $\Delta t < 0$.

Definition

The value defined by the relation:

$$\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} \tag{2.17}$$

is called the **derivative of the vector v** (ordinary derivative, as opposed to directional derivative, which will be defined later) with respect to the scalar t, if such a limit exists. This value will be denoted shortly by $\mathbf{v}'$.

The derivative will be symbolically denoted by $\frac{d\mathbf{v}}{dt}$, and thus

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t}. \tag{2.18}$$

Definition

It is said that a vector function is **differentiable** in point t, if the first derivative in that point exists, i.e. if the following limit value exists

$$\mathbf{v}' = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t}. \tag{2.19}$$

A differentiable function is also continuous. The reverse is not true (see Fig. 4.30).

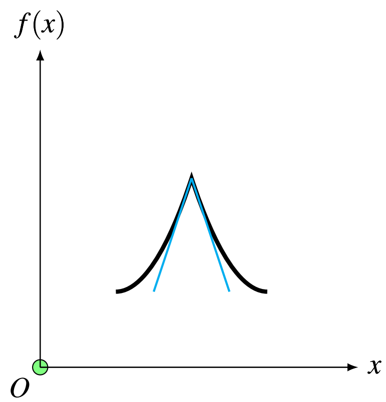


Figure 2.3: An example of a continuous non-differentiable function.

2.1.5 Properties of the derivative

Some properties of the derivative follow

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}+\mathbf{b}) = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}, \tag{2.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{a}) = m\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{a}, \quad m = m(t), \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\cdot\mathbf{b}, \tag{2.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\times\mathbf{b}. \tag{2.23}$$

2.1.6 Differential of the vector function

Suppose that the geometric increment of the vector function $\mathbf{v}(t)$ can be represented as

$$\Delta\mathbf{v} = \mathbf{v}(t+\Delta t) - \mathbf{v}(t) = \mathbf{L}(t)\Delta t + \varepsilon(\Delta t), \tag{2.24}$$

where $\varepsilon(\Delta t)$, when $\Delta t \to 0$, is a higher order vector infinitesimal with respect to Δt, i.e.

$$\lim_{\Delta t\to 0}\frac{\varepsilon(\Delta t)}{\Delta t} = \mathbf{0}. \tag{2.25}$$

Definition

Differential of the vector function $\mathbf{v}(t)$ is the linear part of the increment of the argument $\mathbf{L}(t)\Delta t$ in the geometric increment of the function. This is symbolically denoted by

$$\mathrm{d}\mathbf{v} = \mathbf{L}(t)\Delta t. \tag{2.26}$$

For sufficiently small values of the increment of the variable $\Delta t = \mathrm{d}t$, the geometric increment of the function $\mathbf{v}(t)$, can be approximated by its differential, i.e.

$$\mathbf{v}(t+\Delta t) - \mathbf{v}(t) \approx \mathbf{L}(t)\Delta t, \tag{2.27}$$

that is

$$\mathbf{v}(t+\Delta t) - \mathbf{v}(t) \approx \mathrm{d}\mathbf{v}. \tag{2.28}$$

It follows from (2.24) that

$$\frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \mathbf{L}(t) + \frac{\varepsilon(\Delta t)}{\Delta t}, \tag{2.29}$$

namely, according to (2.25) and (2.18), we obtain

$$\lim_{\Delta t\to 0}\frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \mathbf{L}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{v}'. \tag{2.30}$$

Now (2.26) becomes

$$\mathrm{d}\mathbf{v} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\Delta t = \mathbf{v}'\cdot\mathrm{d}t. \tag{2.31}$$

To obtain the last relation (2.31) we used the definition of the derivation (2.17) and the assumption that $\Delta t = \mathrm{d}t$.

Thus, the differential $\mathrm{d}\mathbf{v}$ is a vector whose direction is tangent to the hodograph.

2.1.7 Higher order derivatives and differentials

Since the derivative of a vector function is also a vector function (of the same variable), a derivative of that function can subsequently be determined, and thus a second derivative obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}') = \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} = \mathbf{v}''. \tag{2.32}$$

In the same way, higher derivatives can also be obtained. Since by definition $\mathbf{v}^{(0)} = \mathbf{v}$, the n–th derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^{n-1}\mathbf{v}}{\mathrm{d}t^{n-1}}\right) = \frac{\mathrm{d}^n\mathbf{v}}{\mathrm{d}t^n} = \mathbf{v}^{(n)}, \tag{2.33}$$

where this relation is true for each $n \in \mathbb{N}$.

The n–th differential of the vector function is, by analogy with (2.31), a product of the n–th derivative and the n–th degree of the differential $\mathrm{d}t$, i.e.

$$\mathrm{d}^n\mathbf{v} = \mathbf{v}^{(n)}\,\mathrm{d}t^n. \tag{2.34}$$

2.1.8 Partial derivative of a vector function of several independent variables

Consider a vector function $\mathbf{v}$, which depends of n scalar variables t_i, $i = 1,2,\cdots,n$, i.e.

$$\mathbf{v} = \mathbf{v}(t_1,t_2,\cdots,t_n) = \mathbf{v}(T). \tag{2.35}$$

T can be perceived as a point in the n–dimensional space, with coordinates t_i $(i = 1,2,\cdots,n)$.

Let us now find the increment of this function, if only one variable, say k, changes, and the other variables remain "frozen", i.e. they are held constant. The increment of the function is

$$\mathbf{v}(t_1,\cdots,t_k+\Delta t_k,\cdots,t_n) - \mathbf{v}(t_1,\cdots,t_k,\cdots,t_n). \tag{2.36}$$

The limit value

Definition

$$\lim_{\Delta t_k \to 0} \frac{\mathbf{v}(t_1,\cdots,t_k+\Delta t_k,\cdots,t_n) - \mathbf{v}(t_1,\cdots,t_k,\cdots,t_n)}{\Delta t_k} = \frac{\partial \mathbf{v}}{\partial t_k} \tag{2.37}$$

is called the **partial derivative** of the vector function $\mathbf{v}$, with respect to the variable t_k, it such a limit value exists.

As this value is also a vector function, of the same variables, partial derivatives of higher order can also be defined, as for example

$$\frac{\partial^2\mathbf{v}}{\partial t_i^2}, \quad \frac{\partial^2\mathbf{v}}{\partial t_i \partial t_j}, \quad \frac{\partial^3\mathbf{v}}{\partial t_i^3}, \quad \frac{\partial^3\mathbf{v}}{\partial t_i^2 \partial t_j}, \quad \frac{\partial^3\mathbf{v}}{\partial t_i \partial t_j \partial t_k}, \cdots \tag{2.38}$$

2.1.9 Differential of a vector function of n scalar variables

Observe the vector function

$$\mathbf{v} = \mathbf{v}(T), \tag{2.39}$$

its point $T(t_1,t_2,\cdots,t_n)$ and some other point $A(a_1,a_2,\cdots,a_n)$. Points A and T belong to the n–dimensional Euclidian space E^n. The increment of the function $\mathbf{v}$ is

$$\Delta\mathbf{v} = \mathbf{v}(T) - \mathbf{v}(A) = \mathbf{v}(t_1,t_2,\cdots,t_n) - \mathbf{v}(a_1,a_2,\cdots,a_n). \tag{2.40}$$

Definition

The function **v=v**(T) is **differentiable** in point A, if its increment can be represented in the form

$$\Delta \mathbf{v} = [\mathbf{p}_1(T)\,(t_1 - a_1) + \cdots + \mathbf{p}_n(T)\,(t_n - a_n)] + \omega(T)\cdot \rho(T,A), \tag{2.41}$$

where $\rho(T,A)$ is the distance between points T and A (in Euclidian space)

$$\rho = \sqrt{(t_1 - a_1)^2 + (t_2 - a_2)^2 + \cdots + (t_n - a_n)^2}, \tag{2.42}$$

and $\omega(T)$ is a continuous function in point A, in which

$$\lim_{T \to A} \omega(T) = \omega(A) = 0. \tag{2.43}$$

Definition

Differential of the vector function $\mathbf{v}(T)$, in point A, is the linear part with respect to the increment of variables $\Delta t_i = t_i - a_i$, in the expression for the increment of the function $\Delta\mathbf{v}$, i.e.

$$\mathrm{d}\mathbf{v}(T,A) = \mathbf{p}_1\,(t_1 - a_1) + \cdots + \mathbf{p}_n\,(t_n - a_n)\,. \tag{2.44}$$

If all variables except the k-th, namely

$$(a_1 = t_1, \cdots, a_{k-1} = t_{k-1}, a_{k+1} = t_{k+1}, \cdots, a_n = t_n),$$

are fixed, then the distance ρ becomes

$$\rho(T,A) = \sqrt{(t_k - a_k)^2} = |t_k - a_k|, \tag{2.45}$$

and the increment of the function takes the form

$$\begin{aligned}\Delta\mathbf{v} = \mathbf{v}(a_1, \cdots, a_{k-1}, t_k, a_{k+1}, \cdots, a_n) - \mathbf{v}(a_1, a_2 \cdots, a_n) = \\ \mathbf{p}_k(t_k - a_k) + \omega(T)\cdot |t_k - a_k|.\end{aligned} \tag{2.46}$$

If the symbol $t_k - a_k = \Delta t_k$ is introduced, it follows from the last equation that

$$\frac{\Delta\mathbf{v}}{\Delta t_k} = \mathbf{p}_k \pm \omega(T). \tag{2.47}$$

and consequently

$$\lim_{\Delta t_k \to 0} \frac{\Delta\mathbf{v}}{\Delta t_k} = \mathbf{p}_k \pm \lim_{T \to A} \omega(T), \tag{2.48}$$

or

$$\mathbf{p}_k = \left.\frac{\partial\mathbf{v}}{\partial t_k}\right|_{T=A}. \tag{2.49}$$

The expression for the differential (2.44) can now take one of the following forms

$$\mathrm{d}\mathbf{v}(T,A) = \frac{\partial\mathbf{v}}{\partial t_1}(t_1 - a_1) + \cdots + \frac{\partial\mathbf{v}}{\partial t_n}(t_n - a_n) \tag{2.50}$$

or, if $t_i - a_i = \Delta t_i$, for $i = 1, 2, \ldots, n$

$$\mathrm{d}\mathbf{v}(T,A) = \frac{\partial \mathbf{v}}{\partial t_1}\Delta t_1 + \cdots + \frac{\partial \mathbf{v}}{\partial t_n}\Delta t_n \tag{2.51}$$

or, for $\Delta t_i = \mathrm{d}t_i$

$$\mathrm{d}\mathbf{v}(T,A) = \frac{\partial \mathbf{v}}{\partial t_1}\mathrm{d}t_1 + \cdots + \frac{\partial \mathbf{v}}{\partial t_n}\mathrm{d}t_n. \tag{2.52}$$

Definition

The expression (2.52) is called the **total differential** of the function $\mathbf{v}$, and the expression

$$\frac{\partial \mathbf{v}}{\partial t_i}\mathrm{d}t_i \quad (i = 1, 2, \cdots, n) \tag{2.53}$$

the **partial differential**.

2.2 Integration

2.2.1 Indefinite integral of a vector function

In the previous section, vector function differentiation was defined. Namely, for a vector function **a** its derivative can be found following the definition (2.17). However, is often necessary to solve the reverse task, namely to find a vector function if its derivative is given.

Let $\mathbf{a}(t)$ be a continuous vector function of the scalar argument t.

Definition

The **primitive function** of the function $\mathbf{a}(t)$ is the function $\mathbf{b}(t)$ the derivative of which

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t} = \mathbf{a}. \tag{2.54}$$

However, as the derivative of a constant vector is equal to zero, i.e.

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} = \mathbf{0}, \tag{2.55}$$

then if $\mathbf{b}(t)$ is a primitive function of a continuous function $\mathbf{a}(t)$, there exists an unlimited set of primitive functions, each of them differing from $\mathbf{b}(t)$ by a vector constant **c**, i.e.

$$\frac{\mathrm{d}(\mathbf{b}+\mathbf{c})}{\mathrm{d}t} = \mathbf{a}. \tag{2.56}$$

Definition

The **indefinite integral** of a vector function **a** is the set of all its primitive functions, denoted by

$$\int \mathbf{a}\,\mathrm{d}t = \mathbf{b}+\mathbf{c}. \tag{2.57}$$

As **a** can be represented by the relation

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \tag{2.58}$$

the indefinite integral of a vector function amounts to a sum of indefinite integrals of scalar functions

$$\int \mathbf{a}\,\mathrm{d}t = \left(\int a_x\,\mathrm{d}t\right)\mathbf{i} + \left(\int a_y\,\mathrm{d}t\right)\mathbf{j} + \left(\int a_z\,\mathrm{d}t\right)\mathbf{k}. \tag{2.59}$$

2.2.2 Definite integral

Let **a** be a bounded function of the parameter t, on the interval (t_A, t_B). Let that interval be divided into a finite number of parts

$$\Delta t_i = t_i - t_{i-1} \tag{2.60}$$

by the points

$$t_A = t_0 < t_1 < \cdots < t_n = t_B. \tag{2.61}$$

Let us now form a sum (usually called the integral sum)

$$\mathbf{I} = \sum_{i=1}^{n} \mathbf{a}(\tau_i)(t_i - t_{i-1}), \tag{2.62}$$

where $\tau_i \in (t_{i-1}, t_i)$.

Definition

If there exists a limit value of the sum $\mathbf{I}$, when n grows indefinitely, where for an arbitrary division of the interval (t_0, t_n) the largest of the parts Δt_i tends to zero, then this limit value is called the **finite integral** (in the Riemannian sense) of the function $\mathbf{a}$

$$\lim_{max|\Delta t_i| \to 0} \sum_{i=1}^{n} \mathbf{a}(\tau_i)\Delta t_i = \int_{t_A}^{t_B} \mathbf{a}\, dt. \tag{2.63}$$

R Note that in addition to Riemann[1] integral, there also exist Stieltjes, Lebesgue and other integrals.

According to the Newton[2]–Leibniz[3] relation

$$\int_{t_A}^{t_B} \mathbf{a}\, dt = \mathbf{b}\,\Big|_{t_A}^{t_B} = \mathbf{b}(t_B) - \mathbf{b}(t_A), \tag{2.64}$$

if $\mathbf{b}$ is a primitive function of the function $\mathbf{a}$.

2.2.3 The line integral of a vector function

In previously defined integrals the integration area could be interpreted as a straight line or its part. However, a natural generalization for vector functions, similar to scalar functions, is to extend the integration to curved lines, surfaces, and volumes.

Curve orientation

Consider a bounded curve in space, given by the vector equation

$$\mathbf{r} = \mathbf{r}(t), \quad t \in [t_A, t_B]. \tag{2.65}$$

To orient the curve means to determine which of the two arbitrary points from the curve is the preceding and which one is the following.

---

[1] Bernhard Riemann (1826-1866), famous German mathematician. He has made significant contributions in geometry, analysis, differential equation theory and number theory.

[2] Sir Isaac Newton (1642–1727), famous English physicist and mathematician. Introduced differential and integral calculus simultaneously with Leibniz (although independently of one another). He formulated many basic laws and methods of investigating problems in physics, using mathematical analysis. His book *Mathematical Principles of Natural Philosophy, 1687* represents a remarkable contribution to classical mechanics. His work is of great importance for both mathematics and physics.

[3] Gottfried Wilhelm Leibniz (1646–1716), German mathematician and philosopher. Introduced differential and integral calculus simultaneously with Newton.

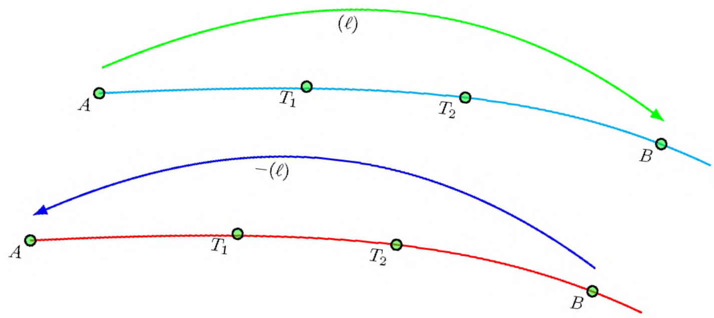


Figure 2.4: Curve orientation.

Consider two different points, T_1 and T_2 from the curve l, determined by the parameter values t_1 and t_2 $(t_1 < t_2)$. If we assume that point T_1 is the preceding and T_2 the following, then an orientation is obtained (Fig. 2.4a). However, the opposite can also be assumed, namely that T_2 is the preceding and T_1 the following ($t_2 < t_1$), which yields another orientation (Fig. 2.4b). Orientation of closed curve, where $\mathbf{r}(t_A) = \mathbf{r}(t_B)$, which means that points A and B, corresponding to parameter values t_A and t_B, respectively, overlap, is determined by observing three points from the curve and selecting a curve orientation in one of the two ways shown in figure 2.5. The order ABC defines one orientation, while the order ACB defines another, opposite orientation.

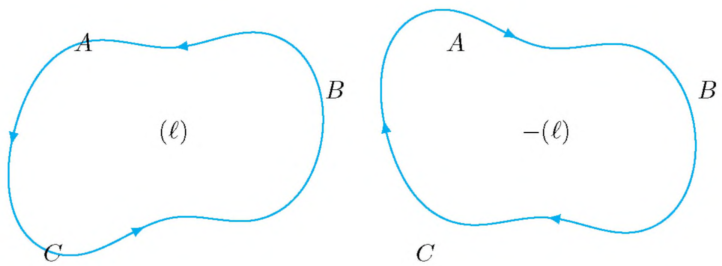


Figure 2.5: Closed curve orientation.

Partition of oriented curve

Partition of oriented curve $(l = AB)$ division assumes that the points of partition

$$A = T_0, T_1, \ldots, T_n = B \tag{2.66}$$

are numerated in order of succession (Fig. 2.6), namely that they correspond to the parameter values

$$t_0, t_1, \ldots, t_n, \tag{2.67}$$

where, for the chosen orientation

$$t_A = t_0 < t_1 < \cdots < t_n = t_B. \tag{2.68}$$

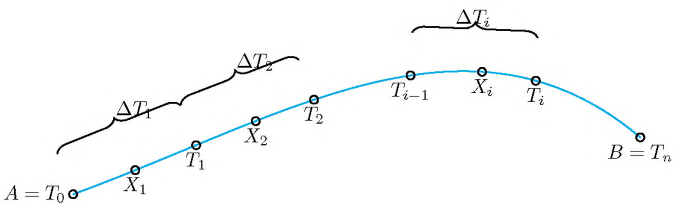


Figure 2.6: Oriented curve division.

Consider now a bounded function $\varphi(\mathbf{r}) = \varphi(X) = \varphi(x,y,z)$, where $\mathbf{r}$ is the position vector of the point $X = X(x,y,z)$. The function φ can be either a scalar or a vector function. Let us partition the oriented curve $l = AB$ by points $T_0, T_1, \ldots, T_n$ into intervals $\Delta T_i = (T_{i-1}, T_i)$, $i = 1, \ldots, n$. Let the point X_i, whose position vector ρ_i, belong to the interval ΔT_i, i.e. $X_i \in \Delta T_i$. Note that in this case i denotes the i–th point, rather than a vector component.

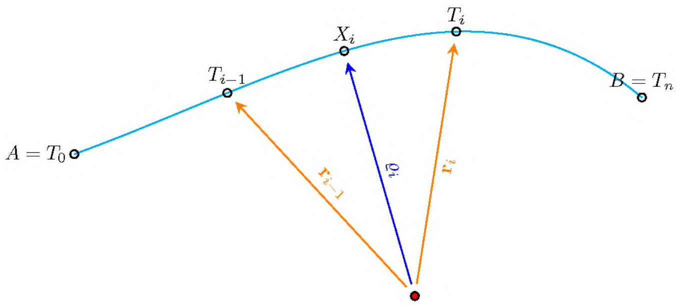


Figure 2.7: Oriented arc AB.

Consider the product

$$\varphi(\rho_i) \circ (\mathbf{r}_i - \mathbf{r}_{i-1}) = \varphi(\rho_i) \circ \Delta\mathbf{r}_i = \varphi(X_i) \circ \Delta\mathbf{r}_i, \tag{2.69}$$

where $\mathbf{r}_i = \mathbf{r}(t_i)$ is the position vector of point T_i. This product can be either a vector or a scalar, depending on the nature of the function φ and the meaning of the $\circ$ symbol for the product (scalar or vector product).

Consider now the sum (**integral sum**)

$$I = \sum_{i=1}^{n} \varphi(X_i) \circ \Delta\mathbf{r}_i. \tag{2.70}$$

Definition

If there exists a limit value of the sum (2.70), when the greatest absolute value $|\Delta\mathbf{r}_i|$ tends to zero, then this value is called the **line integral** of the function $f(X)$ along the curve $l = AB$ and denoted by

$$\lim_{max|\Delta\mathbf{r}_i|\to 0} \sum_{i=1}^{n} \varphi(X_i) \circ \Delta\mathbf{r}_i = \int_l \varphi(\mathbf{r}) \circ d\mathbf{r}. \tag{2.71}$$

Depending on the nature of the function φ and the meaning of the $\circ$ symbol for the product the following three cases are possible

- $\varphi(\mathbf{r})$ is a scalar function, the symbol $\circ$ stands for the multiplication of a vector by a scalar, and the line integral is a vector,
- $\varphi(\mathbf{r})$ is a vector function, the symbol $\circ$ stands for the scalar product, and the line integral is a scalar function,
- $\varphi(\mathbf{r})$ is a vector function, the symbol $\circ$ stands for the vector product, and the line integral is a vector function.

2.2.4 Surface integral

Surface orientation

Definition

A surface S is said to be **orientable** (two-sided) (Fig. 2.8), if it can be partitioned into pieces which can be oriented in such a way that along each curve that is a common boundary of two different pieces, the directions of the curve relative to the two pieces are mutually opposite. If such orientation of the pieces of the surface cannot be established by any division, then the surface is said to be nonorientable (one-sided) (Fig. 2.9)).

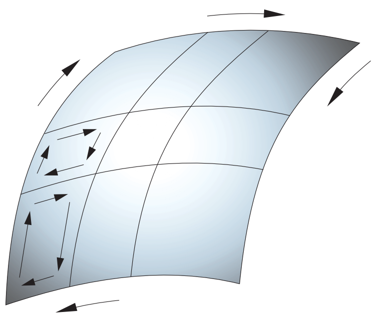


Figure 2.8: Surface orientation.

An example of nonorientable surface is the Möbius[4] strip. It is obtained by twisting a rectangular strip ABCD and joining its end points as follows: A with C and B with D, as shown in Fig. 2.9. If a triangulation (division into triangles) of the ABCD strip is performed and the resulting triangles are oriented (Fig. 2.9c), then when the Möbius strip is formed, the boundary lines AD and BC have the same and not opposite orientations, which makes the Möbius strip a one-sided surface by definition.

[4]Möbius, August Ferdinand (1790–1868), German mathematician. Known for his works in the theory of surfaces, projective geometry and mechanics, as well as number theory.

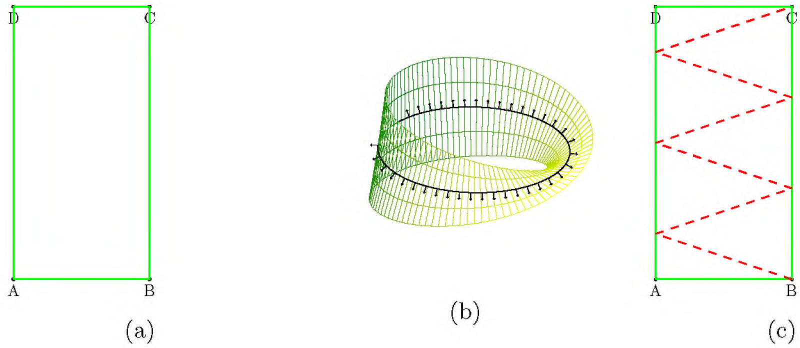


Figure 2.9: Möbius strip

Thus, from the very definition of an orientable surface, it follows that an orientable surface S has two distinct sides. Namely, if a normal line is observed at an arbitrary point in the surface, then two directions can be distinguished: the upward direction if the normal makes an acute angle with the Oz axis and the downward direction if the normal and the axis form an obtuse angle (Fig. 2.10).

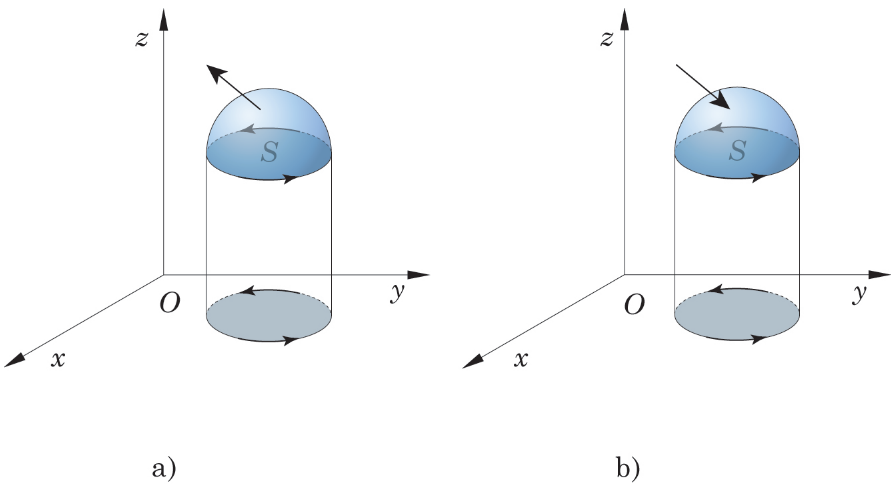


Figure 2.10: Orientable surface.

For the upper part of the surface S the upward direction can be chosen, and for the lower part, the downward direction, in which case the normal is called external.

If the surface is given by the equation $z = f(x,y)$, then the cosine of the angle that the normal forms with the positive part of the Oz-axis, is equal to:

$$\cos\gamma = \begin{cases} \dfrac{1}{\sqrt{1+f_x'^2+f_y'^2}} & \text{—for the upper part} \\ -\dfrac{1}{\sqrt{1+f_x'^2+f_y'^2}} & \text{—for the lower part,} \end{cases} \tag{2.72}$$

where

$$f_x' = \frac{\partial f}{\partial x}, \quad f_y' = \frac{\partial f}{\partial y}.$$

Vector surface integral

Let $\varphi(\mathbf{r}) = \varphi(X) = \varphi(x,y,z)$ be a continuous scalar or vector function of the point $X(x,y,z)$, that is, of the position vector $\mathbf{r}$ of this point. Let us then partition the surface S into a finite number of parts ΔS_i, $i = 1,2,\dots,n$. The area of each part ΔS_i can be represented in vector form, that is, as a vector whose intensity is equal to the area of ΔS_i

$$\Delta \mathbf{S}_i = \Delta S_i \mathbf{n}, \tag{2.73}$$

where $\mathbf{n}$ is the unit vector of the surface normal at an arbitrary point X_i which belongs to ΔS_i.

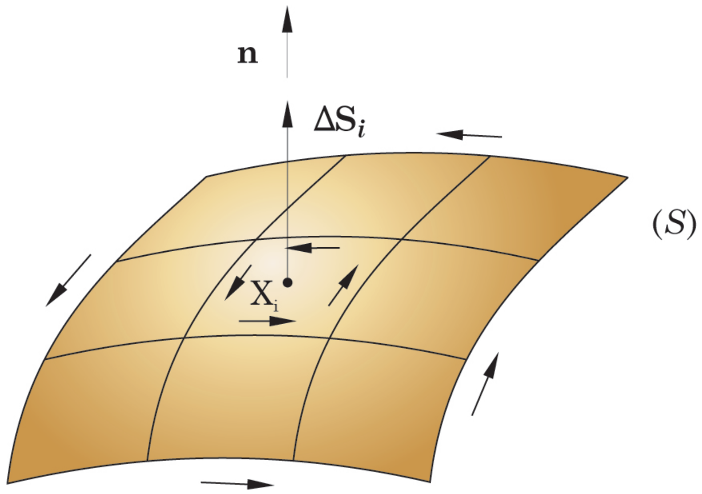


Figure 2.11: Part of the surface ΔS_i.

Let us now form the integral sum

$$I = \sum_{i=1}^{n} \varphi(X_i) \circ \Delta \mathbf{S}_i, \tag{2.74}$$

where $X_i \in \Delta S_i$.

This sum can be a scalar or vector variable, depending on the nature of the function φ and the meaning of the $\circ$ symbol for the product (scalar or vector product).

Definition

The limit value of the sum I, when the largest absolute value $|\Delta S_i|$ tends to zero, is called the **vector surface integral** of the function φ over the surface S and denoted by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \varphi(X_i) \circ \Delta \mathbf{S}_i = \iint_S \varphi(X) \circ d\mathbf{S}, \tag{2.75}$$

if such a limit value exists.

Analysis

Let φ be a vector function, represented in one of the following ways

$$\varphi(X) \equiv \mathbf{v} = v_x \mathrm{i} + v_y \mathrm{j} + v_z \mathrm{k} = [v_x, v_y, v_z], \tag{2.76}$$

and let the $\circ$ symbol stand for the scalar product. The unit vector of the normal can be represented in the form

$$\mathbf{n} = \cos\alpha \mathbf{i} + \cos\beta \mathbf{j} + \cos\gamma \mathbf{k} = [\cos\alpha, \cos\beta, \cos\gamma], \tag{2.77}$$

where α, β and γ are angles the vector $\mathbf{n}$ forms with the coordinate axes Ox, Oy and Oz, respectively. The integral can now be represented in the form

$$\iint_S \varphi(X)\cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{v}\cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S =$$
$$\iint_S (v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma)\,\mathrm{d}S. \tag{2.78}$$

On the other hand, let the equation of the surface S be

$$z = f(x,y), \tag{2.79}$$

then

$$\iint_S v_z\,\mathrm{d}x\mathrm{d}y = \iint_{D_{xy}} v_z \frac{\pm 1}{\sqrt{1+f_x'^2+f_y'^2}}\cdot\left(\pm\sqrt{1+f_x'^2+f_y'^2}\right)\mathrm{d}x\mathrm{d}y =$$
$$= \iint_S v_z \cos\gamma\,\mathrm{d}S, \tag{2.80}$$

where D_{xy} is the projection of the surface S on the plane Oxy, and γ the angle between the normal $\mathbf{n}$ and the z–axis. The sign $\pm$ depends on whether the integration is performed on the upper or the lower part of the surface. A similar result is obtained for the other two integrals

$$\iint_S v_y\,\mathrm{d}z\mathrm{d}x = \iint_S v_y \cos\beta\,\mathrm{d}S, \tag{2.81}$$
$$\iint_S v_x\,\mathrm{d}y\mathrm{d}z = \iint_S v_x \cos\alpha\,\mathrm{d}S. \tag{2.82}$$

By substituting the relations (2.80)-(2.82) in (2.78), we obtain

$$\iint_S v_x\,\mathrm{d}y\mathrm{d}z + v_y\,\mathrm{d}z\mathrm{d}x + v_z\,\mathrm{d}x\mathrm{d}y =$$
$$\iint_S (v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma)\,\mathrm{d}S,$$

the relation between the surface integral with respect to coordinates and surface integral with respect to surface.

3. Examples

3.1 Vector algebra

Problem 2

Prove:

a) commutativity of the sum of vectors.
b) associativity of the sum of vectors.

Proof

a) Consider two vectors **a** and **b** (Fig. 3.1). The common point B is the point where vector **a** ends and vector **b** begins.

Using the parallelogram rule for the sum of vectors, vector **c** is obtained, which starts at point A, and ends at point C, **c=a+b**. Let us now translate the vector **b** so that it starts in point A and vector **a** so that it starts in point D, which is the end point of vector **b**.

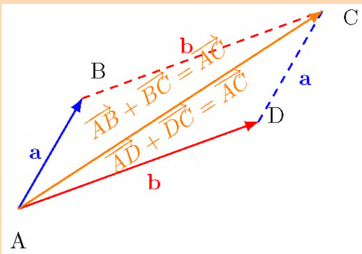


Figure 3.1: Figure in Problem 2.

It is obvious from the figure that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ and $\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}$, that is, **a+b=c** and **b+a=c**. It follows that **a+b=b+a**, namely that the sum of vectors is a commutative operation. The figure depicts a parallelogram $ABCD$ with the diagonal AC, which is why the vector addition rule is called the parallelogram rule.

b) Consider vectors **a**, **b** and **c** (Fig. 3.2). Let us first add vectors **a** and **b** ($OP+PQ=OQ$), and then add vector **c** to the resulting vector (sum) ($OQ+QR=OR$), thus obtaining vector **d**.

Let us now add first vectors **b** and **c** ($\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$), and then add vector **a** to the resulting vector ($\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR}$. We have thus once again obtained the vector **d**, which proves the associativity of the vector sum.

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})=\mathbf{d}.$$

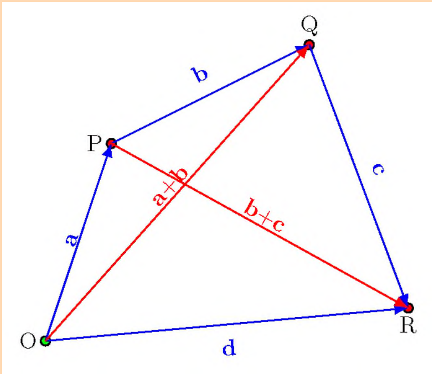


Figure 3.2: Figure in Problem 2.

Problem 3

Prove

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}.$$

Proof

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})=\\ &= a_xb_x\mathbf{i}\times\mathbf{i}+a_xb_y\mathbf{i}\times\mathbf{j}+a_xb_z\mathbf{i}\times\mathbf{k}+a_yb_x\mathbf{j}\times\mathbf{i}+a_yb_y\mathbf{j}\times\mathbf{j}+a_yb_z\mathbf{j}\times\mathbf{k}+\\ &+a_zb_x\mathbf{k}\times\mathbf{i}+a_zb_y\mathbf{k}\times\mathbf{j}+a_zb_z\mathbf{k}\times\mathbf{k}=\\ &=(a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}=\\ &=\text{determinant.}\end{aligned}$$

R <u>Note</u> that this is a "symbolic" determinant. Namely, a determinant yields a scalar value, while in this case the result is a vector. However, the determinant properties are valid for this "determinant" as well.

Problem 4

Prove:

a)

$$\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2>0,\quad \text{except for}\quad \mathbf{a}=\mathbf{0}.$$

b)

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}.$$

c)

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

d)

$$\alpha(\mathbf{a}\cdot\mathbf{b}) = (\alpha\mathbf{a}\cdot\mathbf{b}),$$

where α is a real number.

Proof

a) For $\mathbf{a} = 0$ it is obvious that $\mathbf{a}\cdot\mathbf{a} = 0$. Let $\mathbf{a} \neq 0$, then at least one of the values a_x, a_y or a_z is different from zero. Thus, the following is true

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2 > 0.$$

b)

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta = |\mathbf{b}|\cdot|\mathbf{a}|\cos(-\theta) = \mathbf{b}\cdot\mathbf{a}$$

which proves the commutativity of the scalar product (Fig. 3.4 (a)). We have used here the fact cosine is an even function, namely $\cos\theta = \cos(-\theta)$

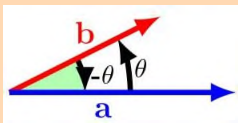


Figure 3.3: Figure in Problem 4.

c) Let $\mathbf{e}_a$ be the unit vector in the direction of vector **a**. The projection of vector (**b+c**) on vector **a** is equal to the sum of projections of vectors **b** and **c** on vector **a** (see Fig. 3.4 (b))

$$(\mathbf{b}+\mathbf{c})\cdot\mathbf{e_a} = \mathbf{b}\cdot\mathbf{e_a}+\mathbf{c}\cdot\mathbf{e_a} \,\Big|\, \cdot|\mathbf{a}|$$

$$(\mathbf{b}+\mathbf{c})\cdot\mathbf{e_a}\cdot|\mathbf{a}| = \mathbf{b}\cdot\mathbf{e_a}\cdot|\mathbf{a}|+\mathbf{c}\cdot\mathbf{e_a}\cdot|\mathbf{a}| =$$
$$= (\mathbf{b}+\mathbf{c})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a} \quad \Rightarrow$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

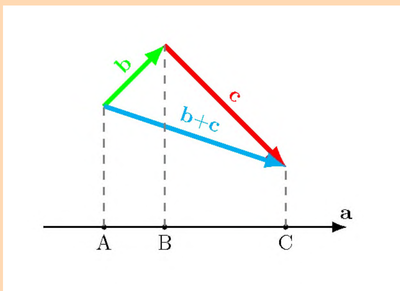


Figure 3.4: Figure in Problem 4.

d)

$$\alpha(\mathbf{a}\cdot\mathbf{b}) = \alpha(a\cdot b\cos\theta) = \alpha a\cdot b\cos\theta = (\alpha a)\cdot b\cos\theta = (\alpha\mathbf{a}\cdot\mathbf{b}).$$

Note that we have used (and we will use in the future) the symbol $|\mathbf{a}| = a$ to denote the magnitude of the vector **a** (similarly for other vectors).

Problem 5

Prove

a) that the vector product is not commutative

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a},$$

b) distributivity of the vector product with respect to vector addition

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

c) distributivity of the vector product with respect to vector addition

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}),$$

d) that the following relation is true

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c},$$

e)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Proof

a) The magnitude of the vector $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ is $ab\sin\theta$ and its direction is determined by the right oriented system (Fig. 3.5).
The magnitude of the vector $\mathbf{b} \times \mathbf{a} = \mathbf{d}$ is $b \cdot a \sin\theta$ and its direction is determined by the right oriented system (Fig. 3.5) b), ie. $\mathbf{d} = \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -\mathbf{c}$.

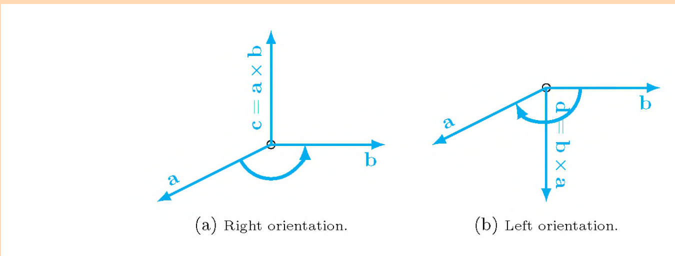


Figure 3.5: Two possible orientations.

Given that vectors **c** and **d** have the same magnitude but opposite directions it follows that

$$\mathbf{c} = -\mathbf{d} \quad \text{that is} \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

Thus, the vector product is not a commutative operation.

b) Let us first prove the distributivity for the case when **a** is normal to the plane formed by vectors **b** and **c** (Fig. 3.6).

Given that **a** is normal to **b**, the vector $\mathbf{a}\times\mathbf{b}$ is normal to the plane formed by vectors **a** and **b** and its magnitude is $a \cdot b \sin 90^\circ = ab$. The same vector is obtained when **b** is multiplied by **a** and then rotated for 90°, as in the picture above.
Similarly, vector $\mathbf{a}\times\mathbf{c}$ is obtained when vector **c** is multiplied by **a** and rotated for 90°, as in Figure 3.6.

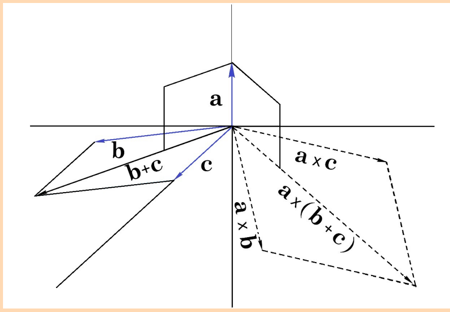


Figure 3.6: Figure in Problem 5.

We obtain the vector $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$ in a similar way by a vector product of vectors $\mathbf{b}+\mathbf{c}$ and vector **a** rotated for 90°, as depicted in Figure 3.6.
Given that $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$ is the diagonal of the parallelogram with sides $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\times\mathbf{c}$, it follows that

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

In the second part of the proof, let us assume that vectors **a**, **b** and **c** can have arbitrary orientations in space (Fig. 3.7).

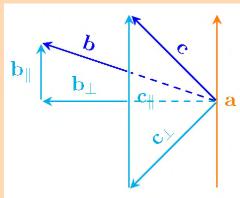


Figure 3.7: Figure in Problem 5.

Let us decompose vectors **b** and **c** into two components, one that is parallel to vector **a** and the other that is normal to **a**. The vector product of components normal to **a** and the vector **a** is given in the first part of the proof, and the vector product of the components parallel to vector **a** and the vector **a** is equal to zero.

c) Given that

$$\mathbf{a} = a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}, \quad \mathbf{b} = b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}, \quad \mathbf{c} = c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}$$

it follows that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix} =$$

$$= (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times[(b_yc_z-b_zc_y)\mathbf{i}+(b_zc_x-b_xc_z)\mathbf{j}+(b_xc_y-b_xc_z)\mathbf{k}] =$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ (b_yc_z-b_zc_y) & (b_zc_x-b_xc_z) & (b_xc_y-b_yc_z)\end{vmatrix} =$$

$$= (a_yb_xc_y-a_yb_yc_x-a_zb_zc_x+a_zb_xc_z)\mathbf{i}+$$
$$+(a_zb_yc_z-a_zb_zc_y-a_xb_xc_y+a_xb_yc_x)\mathbf{j}+$$
$$+(a_xb_zc_x-a_xb_xc_z-a_yb_yc_z+a_yb_zc_y)\mathbf{k}.$$

By grouping the elements with b_x, b_y, b_z, and with c_x, c_y, b_z we obtain:

$$\begin{aligned} &b_x(a_yc_y+a_zc_z)\mathbf{i}-c_x(a_yb_y+a_zb_z)\mathbf{i}+ \\ &+b_y(a_xc_x+a_zc_z)\mathbf{j}-c_y(a_xb_x+a_zb_z)\mathbf{j}+ \\ &+b_z(a_xc_x+a_yc_y)\mathbf{k}-c_z(a_xb_x+a_yb_y)\mathbf{k} \end{aligned}$$

If elements $a_xb_xc_x\mathbf{i}$, $a_yb_yc_y\mathbf{j}$ and $a_zb_zc_z\mathbf{k}$ are both added to and subtracted from the previous expression, we obtain

$$\begin{aligned} &b_x(a_yc_y+a_zc_z)\mathbf{i}-c_x(a_yb_y+a_zb_z)\mathbf{i}+a_xb_xc_x\mathbf{i}-a_xb_xc_x\mathbf{i}+ \\ &+b_y(a_xc_x+a_zc_z)\mathbf{j}-c_y(a_xb_x+a_zb_z)\mathbf{j}+a_yb_yc_y\mathbf{j}-a_yb_yc_y\mathbf{j}+ \\ &+b_z(a_xc_x+a_yc_y)\mathbf{k}-c_z(a_xb_x+a_yb_y)\mathbf{k}+a_zb_zc_z\mathbf{k}-a_zb_zc_z\mathbf{k}, \end{aligned}$$

that is

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =& b_x(\mathbf{a}\cdot\mathbf{c})\mathbf{i}-c_x(\mathbf{a}\cdot\mathbf{b})\mathbf{i}+ \\ &+b_y(\mathbf{a}\cdot\mathbf{c})\mathbf{j}-c_y(\mathbf{a}\cdot\mathbf{b})\mathbf{j}+ \\ &b_z(\mathbf{a}\cdot\mathbf{c})\mathbf{k}-c_z(\mathbf{a}\cdot\mathbf{b})\mathbf{k}. \end{aligned}$$

By extracting the common elements and using (1.44), on p.31, we obtain the relation

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{3.1}$$

d) Given the anticommutativity of the vector product (1.23), it follows that

$$\begin{aligned} (\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(-\mathbf{c}\cdot\mathbf{b})-\mathbf{b}(-\mathbf{c}\cdot\mathbf{a})= \\ =\mathbf{b}(\mathbf{c}\cdot\mathbf{a})-\mathbf{a}(\mathbf{c}\cdot\mathbf{b}). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) it follows that the associative law does not hold for the vector product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}.$$

Problem 6

Prove $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})^2$.

Solution

By using the previous example we obtain

$$\mathbf{x}\times(\mathbf{c}\times\mathbf{a})=\mathbf{c}(\mathbf{x}\cdot\mathbf{a})-\mathbf{a}(\mathbf{x}\cdot\mathbf{c})$$

if $\mathbf{x}=\mathbf{b}\times\mathbf{c}$

$$(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=\mathbf{c}(\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})-\mathbf{a}(\mathbf{b}\times\mathbf{c}\cdot\mathbf{c})=\mathbf{c}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})-\mathbf{a}(\mathbf{b}\cdot\mathbf{c}\times\mathbf{c})=\mathbf{c}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})=(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})^2.$$

Problem 7

Prove

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{c}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{d}) - \mathbf{d}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}\times\mathbf{d}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}\times\mathbf{d})$$

Solution

From $\mathbf{x}\times(\mathbf{c}\times\mathbf{d}) = \mathbf{c}(\mathbf{x}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{x}\cdot\mathbf{c})$ where $\mathbf{x} = \mathbf{a}\times\mathbf{b}$, it follows that

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{c}(\mathbf{a}\times\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}) = \mathbf{c}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{d}) - \mathbf{d}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}).$$

And from $(\mathbf{a}\times\mathbf{b})\times\mathbf{y} = \mathbf{b}(\mathbf{a}\cdot\mathbf{y}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{y})$ where $\mathbf{y} = \mathbf{c}\times\mathbf{d}$, it follows that

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}\times\mathbf{d}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}\times\mathbf{d}).$$

Problem 8

Prove that the following equality is true

$$[\mathbf{a},\mathbf{b},\mathbf{c}]\,\mathbf{d} + [\mathbf{a},\mathbf{c},\mathbf{d}]\,\mathbf{b} - [\mathbf{a},\mathbf{b},\mathbf{d}]\,\mathbf{c} - [\mathbf{b},\mathbf{c},\mathbf{d}]\,\mathbf{a} = 0.$$

Solution

Observe first the vector product of two vector products

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})\,.$$

This product yields a vector normal to both vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. It follows that this vector lies in the plane determined by vectors **a**, **b**, but also in the plane determined by vectors **c**, **d**, and it can thus be represented in two ways

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{a},\mathbf{c},\mathbf{d}]\,\mathbf{b} - [\mathbf{b},\mathbf{c},\mathbf{d}]\,\mathbf{a},$$
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{a},\mathbf{b},\mathbf{d}]\,\mathbf{c} - [\mathbf{a},\mathbf{b},\mathbf{c}]\,\mathbf{d}.$$

By subtracting these equations we obtain

$$[\mathbf{a},\mathbf{b},\mathbf{c}]\,\mathbf{d} + [\mathbf{a},\mathbf{c},\mathbf{d}]\,\mathbf{b} - [\mathbf{a},\mathbf{b},\mathbf{d}]\,\mathbf{c} - [\mathbf{b},\mathbf{c},\mathbf{d}]\,\mathbf{a} = 0.$$

Problem 9

Prove that three orthogonal vectors in a 3–D space are linearly independent.

Proof

Assume that vectors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{p}$ are mutually orthogonal, namely that $\mathbf{n}\cdot\mathbf{m} = 0$, $\mathbf{n}\cdot\mathbf{p} = 0$ and $\mathbf{m}\cdot\mathbf{p} = 0$. Since the space is three-dimensional a fourth vector that would be mutually orthogonal with these three vectors does not exist.

As per definition, vectors are linearly independent if the following is true

$$\alpha\mathbf{n} + \beta\mathbf{m} + \gamma\mathbf{p} = 0 \Rightarrow \alpha = \beta = \gamma = 0.$$

From here, it follows that

$$\begin{aligned}
\alpha\mathbf{n} + \beta\mathbf{m} + \gamma\mathbf{p} = 0/\cdot\mathbf{n} \Rightarrow &\quad \alpha\mathbf{n}\cdot\mathbf{n} + \beta\mathbf{m}\cdot\mathbf{n} + \gamma\mathbf{p}\cdot\mathbf{n} = 0 \Rightarrow \\
\Rightarrow &\quad \alpha|\mathbf{n}|^2 + 0 + 0 = 0 \Rightarrow \underline{\alpha = 0}, \\
\alpha\mathbf{n} + \beta\mathbf{m} + \gamma\mathbf{p} = 0/\cdot\mathbf{m} \Rightarrow &\quad \alpha\mathbf{n}\cdot\mathbf{m} + \beta\mathbf{m}\cdot\mathbf{m} + \gamma\mathbf{p}\cdot\mathbf{m} = 0 \Rightarrow \\
\Rightarrow &\quad 0 + \beta|\mathbf{m}|^2 + 0 = 0 \Rightarrow \underline{\beta = 0}, \\
\alpha\mathbf{n} + \beta\mathbf{m} + \gamma\mathbf{p} = 0/\cdot\mathbf{p} \Rightarrow &\quad \alpha\mathbf{n}\cdot\mathbf{p} + \beta\mathbf{m}\cdot\mathbf{p} + \gamma\mathbf{p}\cdot\mathbf{p} = 0 \Rightarrow \\
\Rightarrow &\quad 0 + 0 + \gamma|\mathbf{p}|^2 = 0 \Rightarrow \underline{\gamma = 0}.
\end{aligned}$$

Problem 10

Give a geometric interpretation of the vector product.

Answer

a) The area of the parallelogram constructed over vectors $\mathbf{a}$ and $\mathbf{b}$ (Fig. 3.8), is

$$P = |\mathbf{b}|\cdot h = |\mathbf{b}|\cdot|\mathbf{a}|\sin\theta,$$

that is

$$P = |\mathbf{a}\times\mathbf{b}|.$$

Thus, the magnitude of a vector product of two vectors is equal to the area of the parallelogram formed by these two vectors.

b) The area of a triangle whose two sides are vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to half the intensity of their vector product (Fig. 3.8).

$$P_\Delta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|.$$

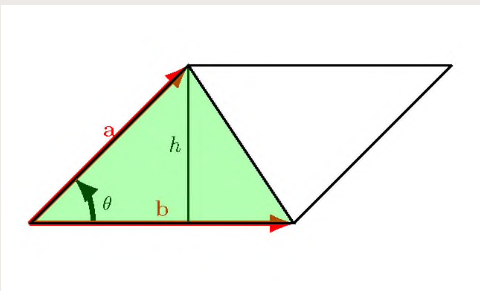


Figure 3.8: Area of a triangle.

Problem 11

Give a geometric interpretation of the mixed product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

Answer

Consider a parallelepiped (Fig. 3.9) constructed over vectors **a**, **b** and **c**. Its volume is $V = h\cdot P$, where h is the height, a P the area of the base. Given that the base is a parallelogram constructed over vectors **b** and **c**, according to the previous example it follows that

$$P = |\mathbf{b}\times\mathbf{c}|.$$

Let **n** be a unit vector with direction orthogonal to the plane of the base (defined by vectors **b** and **c**). The height of the parallelepiped, which corresponds to the base constructed over the vectors **b** and **c**, is equal to

$$h = \mathbf{a}\cdot\mathbf{n} \qquad (\mathbf{a}\cdot\mathbf{n} = a\cdot 1\cos\alpha, \quad \text{where } \alpha \text{ is the angle between vectors } \mathbf{a} \text{ and } \mathbf{n}),$$

and thus it follows that

$$V = h\cdot P = \mathbf{a}\cdot\mathbf{n}|\mathbf{b}\times\mathbf{c}|.$$

Given that

$$\mathbf{b}\times\mathbf{c} = \mathbf{n}|\mathbf{b}\times\mathbf{c}|$$

we finally obtain

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|. \qquad (3.3)$$

Thus the absolute value of the mixed product of three vectors is equal to the volume of the parallelepiped formed by these vectors (the volume is equal to the absolute value, as it cannot be negative).

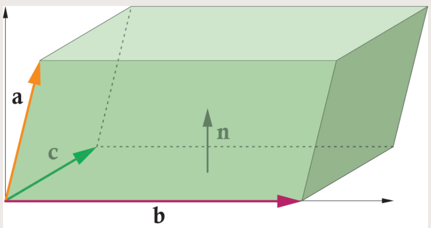


Figure 3.9: Volume of the parallelepiped.

Problem 12

Prove that **a=b** $\Leftrightarrow a_x = b_x, \quad a_y = b_y, \quad a_z = b_z.$

Problem 13

Prove that

$$\alpha\mathbf{a} = [\alpha a_x, \alpha a_y, \alpha a_z].$$

Proof

$$\alpha\mathbf{a} = \alpha(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) = \alpha a_x\mathbf{i} + \alpha a_y\mathbf{j} + \alpha a_z\mathbf{k}.$$

Problem 14

Prove that

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z.$$

Proof

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) = \\ &= a_xb_x\mathbf{i}\cdot\mathbf{i} + a_xb_y\mathbf{i}\cdot\mathbf{j} + a_xb_z\mathbf{i}\cdot\mathbf{k} + a_yb_x\mathbf{j}\cdot\mathbf{i} + a_yb_y\mathbf{j}\cdot\mathbf{j} + a_yb_z\mathbf{j}\cdot\mathbf{k} + \\ &+ a_zb_x\mathbf{k}\cdot\mathbf{i} + a_zb_y\mathbf{k}\cdot\mathbf{j} + a_zb_z\mathbf{k}\cdot\mathbf{k} = \\ &= a_xb_x + a_yb_y + a_zb_z.\end{aligned}$$

Problem 15

Prove that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Proof

Given that

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot((b_yc_z - b_zc_y)\mathbf{i} + (b_zc_x - b_xc_z)\mathbf{j} + (b_xc_y - b_yc_x)\mathbf{k}) = \\ &= a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - b_yc_x) = determinant.\end{aligned}$$

In the proof we used

$$\mathbf{i}\times\mathbf{i} = \mathbf{0}, \quad \mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}, \quad \mathbf{j}\times\mathbf{j} = \mathbf{0}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{j}\times\mathbf{i} = -\mathbf{k},$$
$$\mathbf{k}\times\mathbf{k} = \mathbf{0}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}, \quad \mathbf{k}\times\mathbf{j} = -\mathbf{i}.$$

Problem 16

Prove that

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2}.$$

Proof

Given that the square of the magnitude of the vector product, according to definition (1.19), is equal to

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\gamma,$$

and the scalar product of vectors **a** and **b**, according to definition (1.12), is equal to

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\gamma \quad \text{i} \quad |\mathbf{a}|\,|\mathbf{a}| = |\mathbf{a}|^2,$$

we finally obtain

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\gamma = |\mathbf{a}|^2\,|\mathbf{b}|^2\left(1-\cos^2\gamma\right) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2.$$

Problem 17

Prove that all vectors, which are linearly dependent of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, lie in the plane OM_1M_2, if point O is the start, and M_1 and M_2 are the end of vector $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively.

Proof

Two arbitrary scalars α and β define a vector **a** that is the linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

$$\mathbf{a} = \alpha\mathbf{a_1} + \beta\mathbf{a_2}.$$

Take a vector **b** perpendicular to the plane OM_1M_2. Given that $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in that plane it follows that $\mathbf{b}\cdot\mathbf{a}_1 = 0$ and $\mathbf{b}\cdot\mathbf{a}_2 = 0$. Observe now the following scalar product

$$\mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot(\alpha\mathbf{a}_1 + \beta\mathbf{a}_2) = \mathbf{b}\cdot\alpha\mathbf{a}_1 + \mathbf{b}\cdot\beta\mathbf{a}_2 = \alpha\mathbf{b}\cdot\mathbf{a}_1 + \beta\mathbf{b}\cdot\mathbf{a}_2 = \alpha 0 + \beta 0 = 0.$$

Given that it is also equal to zero, it follows that the vector **b** is normal to the vector **a** and that consequently vector **a** lies in the plane OM_1M_2.

Problem 18

Prove that any four vectors in a 3–D space are linearly dependent.

Problem 19

Prove that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{0}$ is the necessary and sufficient condition for the vectors $\mathbf{a},\mathbf{b}$ and **c** to lie in the same plane (to be coplanar).

Proof

The condition is necessary. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are in the same plane, then the volume of the parallelepiped they form is equal to zero. From there, it follows that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{0}$ (see example 11 on p. 63).

The condition is sufficient. If $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{0}$ then the volume of the parallelepiped formed by these three vectors is equal to zero, and thus it follows that they lie in the same plane, i.e. they are coplanar.

Problem 20

Let $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}\neq 0$ and $\mathbf{a}'=\dfrac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$ $\quad \mathbf{b}'=\dfrac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$ $\quad \mathbf{c}'=\dfrac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$. Prove

a) $\mathbf{a}'\cdot\mathbf{a}=\mathbf{b}'\cdot\mathbf{b}=\mathbf{c}'\cdot\mathbf{c}=1$,

b) $\mathbf{a}'\cdot\mathbf{b}=\mathbf{a}'\cdot\mathbf{c}=0, \quad \mathbf{b}'\cdot\mathbf{a}=\mathbf{b}'\cdot\mathbf{c}=0, \quad \mathbf{c}'\cdot\mathbf{a}=\mathbf{c}'\cdot\mathbf{b}=0$,

c) $\mathbf{a}'\cdot\mathbf{b}'\times\mathbf{c}'=\dfrac{1}{V}$, if $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=V$.

d) $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are not coplanar, given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar (initial assumption).

Solution

a)

$$\mathbf{a}'\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}'=\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=1.$$

$$\mathbf{b}'\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{b}'=\mathbf{b}\cdot\frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=1.$$

$$\mathbf{c}'\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{c}'=\mathbf{c}\cdot\frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=1.$$

b)

$$\mathbf{a}'\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}'=\mathbf{b}\cdot\frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{b}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{c}\cdot\mathbf{b}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=0,$$

$$\mathbf{a}'\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}'=\frac{\mathbf{c}\cdot\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\mathbf{c}\cdot\frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=\frac{\mathbf{b}\cdot\mathbf{c}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}=0.$$

The same can be proved for other vector products analogously.

The same result follows from the fact that vector $\mathbf{a}'$ has the same direction as the vector product $\mathbf{b}\times\mathbf{c}$ and is therefore perpendicular to the plane formed by these two vectors $\mathbf{a}'\cdot(\alpha\mathbf{b}+\beta\mathbf{c})=0$.

Note that, according to the definition (1.25), sets of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ form a reciprocal or conjugate system of vectors.

c) Given that, according to the assumption

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=V,$$

it follows that

$$\mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}')=\frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\cdot\left(\frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\times\frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\right)=$$

$$=\frac{1}{V^3}(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})].$$

Using the property of the triple product (example 5d, p. 58) we obtain:

$$\mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}') = \frac{1}{V^3}(\mathbf{b}\times\mathbf{c})\cdot\{[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}]\mathbf{a}-[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}]\mathbf{c}\} =$$
$$= \frac{1}{V^3}(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\mathbf{a} = \frac{1}{V^3}[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]^2 = \frac{1}{V^3}V^2 = \frac{1}{V}.$$

d) Vectors **a**, **b** and **c** are not coplanar, given that their mixed product is not equal to zero, and therefore its reciprocal is also different from zero. Given that, according to example (20c, p. 66), the value of this reciprocal is equal to the mixed product of vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, it follows that these vectors are not coplanar (see example 19, p. 65).

Problem 21

Prove that the vectors of one system can be uniquely determined by the vectors of the reciprocal system of vectors and that these relations are reciprocal.

Proof

Let **a**, **b** and **c** be linearly independent vectors in a 3-D space. Thus, an arbitrary vector in this space can be represented as their linear combination

$$\mathbf{d} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}.$$

Coefficients α, β i γ can be determined as follows. By multiplying both sides of the previous relation by the vector $\mathbf{b}\times\mathbf{c}$ we obtain

$$\mathbf{d}\cdot\mathbf{b}\times\mathbf{c} = \alpha\mathbf{a}\cdot\mathbf{b}\times\mathbf{c},$$

given that $\mathbf{b}\cdot\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\cdot\mathbf{b}\times\mathbf{c}$ are both equal to zero, being mixed products of coplanar vectors. By introducing

$$V = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c},$$

we obtain

$$\alpha = \frac{\mathbf{d}\cdot\mathbf{b}\times\mathbf{c}}{V}.$$

It can analogously be proved that

$$\beta = \frac{\mathbf{d}\cdot\mathbf{c}\times\mathbf{a}}{V}, \quad \gamma = \frac{\mathbf{d}\cdot\mathbf{a}\times\mathbf{b}}{V}.$$

Finally, we obtain

$$\mathbf{d} = \frac{(\mathbf{d}\cdot\mathbf{b}\times\mathbf{c})}{V}\mathbf{a}+\frac{(\mathbf{d}\cdot\mathbf{c}\times\mathbf{a})}{V}\mathbf{b}+\frac{(\mathbf{d}\cdot\mathbf{a}\times\mathbf{b})}{V}\mathbf{c}.$$

According to the previous example, by introducing reciprocal vectors using the same relations, we obtain

$$\mathbf{d} = (\mathbf{d}\cdot\mathbf{a}')\,\mathbf{a}+(\mathbf{d}\cdot\mathbf{b}')\,\mathbf{b}+(\mathbf{d}\cdot\mathbf{c}')\,\mathbf{c}.$$

Problem 22

Determine vector $\mathbf{r}$ if its scalar products ($a_i = \mathbf{r}\cdot\mathbf{A}_i$) with three noncoplanar vectors $\mathbf{A}_i$ ($i = 1,2,3$) are given.

Solution

The following equations

$$\mathbf{A}_1' = \frac{\mathbf{A}_2\times\mathbf{A}_3}{V}, \quad \mathbf{A}_2' = \frac{\mathbf{A}_3\times\mathbf{A}_1}{V}, \quad \mathbf{A}_3' = \frac{\mathbf{A}_1\times\mathbf{A}_2}{V}. \tag{3.4}$$

determine the reciprocal system of vectors $\mathbf{A_i'}$ ($i = 1,2,3$), where $V = \mathbf{A_1}\cdot\mathbf{A_2}\times\mathbf{A_3}$. Let us decompose vector $\mathbf{r}$ in the directions of three noncoplanar vectors $\mathbf{A}_i'$ using the Gibbs[1] formula

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{A}_1)\mathbf{A}_1' + (\mathbf{r}\cdot\mathbf{A}_2)\mathbf{A}_2' + (\mathbf{r}\cdot\mathbf{A}_3)\mathbf{A}_3' = \sum_{i=1}^{3}(\mathbf{r}\cdot\mathbf{A}_i)\mathbf{A}_i'.$$

Given that, according to the initial assumption $a_i = \mathbf{r}\cdot\mathbf{A_i}$, the previous relation becomes

$$\mathbf{r} = \sum_{i=1}^{3} a_i\mathbf{A}_i'.$$

Using the relation for reciprocal vectors (3.4) we finally obtain

$$\mathbf{r} = \frac{a_1(\mathbf{A}_2\times\mathbf{A}_3) + a_2(\mathbf{A}_3\times\mathbf{A}_1) + a_3(\mathbf{A}_1\times\mathbf{A}_2)}{V}.$$

Problem 23

Prove that, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are noncoplanar vectors and $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$, then $\alpha = \beta = \gamma = 0$.

Proof

Assume that $\alpha \neq 0$. It follows from $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$ that $\alpha\mathbf{a} = -\beta\mathbf{b} - \gamma\mathbf{c}$, that is, $\mathbf{a} = -\frac{\beta}{\alpha}\mathbf{b} - \frac{\gamma}{\alpha}\mathbf{c}$. It further follows that vector $\mathbf{a}$ lies in the plane formed by $\mathbf{b}$ i $\mathbf{c}$, which is in contradiction with the initial assumption that the vectors are noncoplanar. Thus, $\alpha = 0$. It can be proved, by analogy, that $\beta = 0$ and $\gamma = 0$.

[1] Josiah Willard Gibbs (1836-1903). American mathematician, one of the founders of vector analysis, mathematical thermodynamics and statistical mechanics (Elementary Principles in Statistical Mechanics, 1902). His work was of great importance for the development of vector analysis and mathematical physics.

3.2 Vector analysis

Problem 24

Prove that two conjugated systems of vectors represent two noncoplanar systems of the same orientation.

Problem 25

If $\mathbf{A}$ and $\mathbf{B}$ are differentiable functions of scalar u, prove that

a)

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\cdot\mathbf{B},$$

b)

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\times\mathbf{B}.$$

Proof

a)

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{A}\cdot\mathbf{B}) = \lim_{\Delta u\to 0}\frac{(\mathbf{A}+\Delta\mathbf{A})\cdot(\mathbf{B}+\Delta\mathbf{B})-\mathbf{A}\cdot\mathbf{B}}{\Delta u} =$$

$$\lim_{\Delta u\to 0}\frac{\mathbf{A}\cdot\Delta\mathbf{B}+\Delta\mathbf{A}\cdot\mathbf{B}+\Delta\mathbf{A}\cdot\Delta\mathbf{B}}{\Delta u} = \lim_{\Delta u\to 0}\left(\mathbf{A}\cdot\frac{\Delta\mathbf{B}}{\Delta u}+\frac{\Delta\mathbf{A}}{\Delta u}\cdot\mathbf{B}+\frac{\Delta\mathbf{A}\cdot\Delta\mathbf{B}}{\Delta u}\right) =$$

$$= \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\cdot\mathbf{B},$$

where we used $\lim_{\Delta u\to 0}(\Delta\mathbf{A}\cdot\Delta\mathbf{B})/\Delta u = 0$.

b)

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{A}\times\mathbf{B}) = \lim_{\Delta u\to 0}\frac{(\mathbf{A}+\Delta\mathbf{A})\times(\mathbf{B}+\Delta\mathbf{B})-\mathbf{A}\times\mathbf{B}}{\Delta u} =$$

$$\lim_{\Delta u\to 0}\frac{\mathbf{A}\times\Delta\mathbf{B}+\Delta\mathbf{A}\times\mathbf{B}+\Delta\mathbf{A}\times\Delta\mathbf{B}}{\Delta u} = \lim_{\Delta u\to 0}\left(\mathbf{A}\times\frac{\Delta\mathbf{B}}{\Delta u}+\frac{\Delta\mathbf{A}}{\Delta u}\times\mathbf{B}+\frac{\Delta\mathbf{A}\times\Delta\mathbf{B}}{\Delta u}\right) =$$

$$= \mathbf{A}\times\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\times\mathbf{B},$$

where we used $\lim_{\Delta u\to 0}(\Delta\mathbf{A}\times\Delta\mathbf{B})/\Delta u = 0$.

Problem 26

If a vector function $\mathbf{A}$ has a constant magnitude ($|\mathbf{A}|$ is independent of the variable t, and the components A_x, A_y and A_z are functions of t) prove that $\mathbf{A}$ and $\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$ are orthogonal if $\left|\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right| \neq 0$.

Solution

Given that $\mathbf{A}$ has a constant magnitude, it follows that $\mathbf{A}\cdot\mathbf{A} = const$. Then $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot\mathbf{A} = 2\mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = 0$. From that, it further follows that $\mathbf{A}$ and $\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$ are orthogonal if $\left|\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right| \neq 0$.

Problem 27

Prove that $\mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = A\cdot\frac{\mathrm{d}A}{\mathrm{d}t}$, where A is a scalar function $A = |\mathbf{A}|$.

Solution

Given that $\mathbf{A}\cdot\mathbf{A} = A^2$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\cdot\mathbf{A}) = \frac{\mathrm{d}}{\mathrm{d}t}(A^2)$, and we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot\mathbf{A} = 2\mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}.$$

On the other hand, given that

$$\frac{\mathrm{d}}{\mathrm{d}t}(A^2) = 2A\frac{\mathrm{d}A}{\mathrm{d}t}$$

from the two last relations it follows that

$$\mathbf{A}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = A\cdot\frac{\mathrm{d}A}{\mathrm{d}t}.$$

Problem 28

If $\mathbf{F}$ is a function of x,y,z,t where x,y,z are functions of t prove that

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \frac{\partial\mathbf{F}}{\partial t} + \frac{\partial\mathbf{F}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\mathbf{F}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\mathbf{F}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

Proof

Assume that

$$\mathbf{F} = F_1(x,y,z,t)\mathbf{i} + F_2(x,y,z,t)\mathbf{j} + F_3(x,y,z,t)\mathbf{k}.$$

Then

$$\begin{gathered}
\mathrm{d}\mathbf{F} = \mathrm{d}F_1\mathbf{i} + \mathrm{d}F_2\mathbf{j} + \mathrm{d}F_3\mathbf{k} = \\
\left[\frac{\partial F_1}{\partial t}\mathrm{d}t + \frac{\partial F_1}{\partial x}\mathrm{d}x + \frac{\partial F_1}{\partial y}\mathrm{d}y + \frac{\partial F_1}{\partial z}\mathrm{d}z\right]\mathbf{i} + \\
\left[\frac{\partial F_2}{\partial t}\mathrm{d}t + \frac{\partial F_2}{\partial x}\mathrm{d}x + \frac{\partial F_2}{\partial y}\mathrm{d}y + \frac{\partial F_2}{\partial z}\mathrm{d}z\right]\mathbf{j} + \\
\left[\frac{\partial F_3}{\partial t}\mathrm{d}t + \frac{\partial F_3}{\partial x}\mathrm{d}x + \frac{\partial F_3}{\partial y}\mathrm{d}y + \frac{\partial F_3}{\partial z}\mathrm{d}z\right]\mathbf{k} = \\
\left(\frac{\partial F_1}{\partial t}\mathbf{i} + \frac{\partial F_2}{\partial t}\mathbf{j} + \frac{\partial F_3}{\partial t}\mathbf{k}\right)\mathrm{d}t + \left(\frac{\partial F_1}{\partial x}\mathbf{i} + \frac{\partial F_2}{\partial x}\mathbf{j} + \frac{\partial F_3}{\partial x}\mathbf{k}\right)\mathrm{d}x + \\
\left(\frac{\partial F_1}{\partial y}\mathbf{i} + \frac{\partial F_2}{\partial y}\mathbf{j} + \frac{\partial F_3}{\partial y}\mathbf{k}\right)\mathrm{d}y + \left(\frac{\partial F_1}{\partial z}\mathbf{i} + \frac{\partial F_2}{\partial z}\mathbf{j} + \frac{\partial F_3}{\partial z}\mathbf{k}\right)\mathrm{d}z = \\
\frac{\partial \mathbf{F}}{\partial t}\mathrm{d}t + \frac{\partial \mathbf{F}}{\partial x}\mathrm{d}x + \frac{\partial \mathbf{F}}{\partial y}\mathrm{d}y + \frac{\partial \mathbf{F}}{\partial z}\mathrm{d}z,
\end{gathered}$$

and we obtain

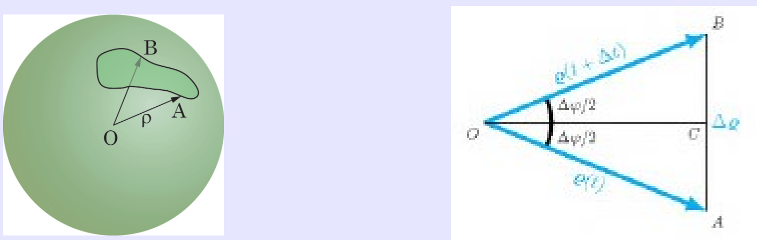
$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mathbf{F}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mathbf{F}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

Problem 29

Find the derivative of a vector with constant magnitude.

Solution

Observe a vector function ρ of the scalar variable t, where $|\rho| = \rho$ =const. The hodograph of this vector is a curve that lies on a sphere with radius ρ (Fig. 3.10a).



(a) (b)

Figure 3.10: Figure in Problem 29.

Given that $\rho = const.$, it follows that $\rho(t+\Delta t) = \rho(t)$, an the triangle $\triangle OAB$ is isosceles (Fig. 3.10b).

The derivative of a vector function is a vector, which can be represented, the same as any other vector, by its magnitude and its unit direction vector

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left|\frac{\mathrm{d}\rho}{\mathrm{d}t}\right| \cdot \mathbf{p}.$$

Let us first determine its magnitude

$$\begin{aligned}\left|\frac{d\rho}{dt}\right| &= \lim_{\Delta t\to 0}\left|\frac{\Delta\rho}{\Delta t}\right| = \lim_{\Delta t\to 0}\frac{|\Delta\rho|}{\Delta t} = \lim_{\Delta t\to 0}\frac{\overline{AB}}{\Delta t} = \\ &= \lim_{\Delta t\to 0}\frac{2\rho\sin(\Delta\varphi/2)}{\Delta t} = 2\rho\lim_{\Delta t\to 0}\frac{\sin(\Delta\varphi/2)}{\Delta\varphi/2}\frac{\Delta\varphi}{2\Delta t} = \\ &= \rho\lim_{\Delta t\to 0}\frac{\sin(\Delta\varphi/2)}{\Delta\varphi/2}\lim_{\Delta t\to 0}\frac{\Delta\varphi}{2\Delta t} = \\ &= \rho\dot{\varphi}.\end{aligned}$$

Here we used the fact that from $\Delta t\to 0$ it follows that $\Delta\varphi\to 0$, that is, $\lim_{\Delta t\to 0}\frac{\sin(\Delta\varphi/2)}{\Delta\varphi/2}=1$. We denoted the derivative of a variable by time t by a point above the variable, which is common in mechanics. Given that $\rho^2=\rho\cdot\rho=\rho^2=\text{const.}$, by deriving by t we obtain

$$2\dot{\rho}\cdot\rho=0,$$

and it thus follows that vectors ρ and $\dot{\rho}$ are orthogonal. Finally, for the derivative of a vector with constant magnitude we obtain

$$\dot{\rho}=|\dot{\rho}|\mathbf{p}=\rho\dot{\varphi}\mathbf{p}=\rho\omega\mathbf{p},$$

where $\rho\perp\mathbf{p}$, and $\omega=\dot{\varphi}$ is the angular velocity, a variable that characterizes the change of the direction of this vector.

Bearing in mind that a vector product of two vectors yields an orthogonal vector, it follows that

$$\dot{\rho}=\omega\times\rho.$$

Given that the magnitude of the vector product $|\dot{\rho}|=\omega\cdot\rho\cdot\sin(\omega,\rho)=\omega\cdot\rho$, it follows that $\sin(\omega,\rho)=1\Rightarrow\angle(\omega,\rho)=\pi/2$, that is, vectors ρ,ω and $\dot{\rho}$ are mutually orthogonal.

Problem 30

Represent the velocity of a point of a rigid body as it rotates about a fixed point.

Solution

Observe an arbitrary point of the body M, whose position vector is $\mathbf{r}$. The body rotates about the fixed point C, whose position vector $\mathbf{r}_C$ is constant. If we denote vector $\overrightarrow{CM}$ by ρ, then (see Figure 3.11)

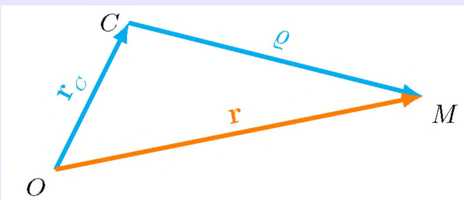


Figure 3.11: Position of a point of a rigid body.

$$\mathbf{r} = \mathbf{r}_c + \rho \quad \Rightarrow \quad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}_c}{dt} + \frac{d\rho}{dt} = \frac{d\rho}{dt}.$$

We used here the fact that the vector $\mathbf{r}_C$ is constant (the point C does not change its position), and thus its derivative is equal to zero.

Given that the body is rigid[2], it follows that the magnitude of the vector $|\rho|$ =const, and thus, according to the previous example

$$\mathbf{v} = \frac{d\rho}{dt} = \omega \times \rho = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \mathbf{i}(\omega_y z - \omega_z y) + \mathbf{j}(\omega_z x - \omega_x z) + \mathbf{k}(\omega_x y - \omega_y x).$$

We denoted here the angular velocity of the body by ω.

Problem 31

Prove the Frenet-Serret [3]formulas

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \frac{1}{\rho}\mathbf{n} = (k_1\mathbf{n}), \\ \frac{d\mathbf{n}}{ds} &= -k_1\mathbf{t} + k_2\mathbf{b} = -\frac{1}{\rho}\mathbf{t} + \frac{1}{T}\mathbf{b}, \\ \frac{d\mathbf{b}}{ds} &= -k_2\mathbf{n} = -\frac{1}{T}\mathbf{n}, \end{aligned} \tag{3.5}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of a natural trihedron, ρ is the radius of curvature, and T is the radius of torsion.

Solution

Observe the natural trihedron formed by the unit vectors of the tangent $\mathbf{t}$, main normal $\mathbf{n}$ and binormal $\mathbf{b}$ (see Figure 3.12). These three vectors form a right trihedron, and thus

$$\begin{aligned} \mathbf{t} \times \mathbf{n} &= \mathbf{b}, \\ \mathbf{n} \times \mathbf{b} &= \mathbf{t}, \\ \mathbf{b} \times \mathbf{t} &= \mathbf{n}. \end{aligned}$$

[2]The term rigid body implies that the distance between any two points of the body, while moving, remains unchanged, that is, $|\overrightarrow{CM}| = const$.

[3]Jean-Frédéric Frenet (1816-1900) and Joseph Alfred Serret (1819-1885), French mathematicians.

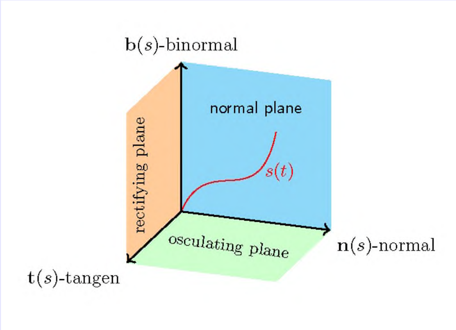


Figure 3.12: Natural trihedron.

The space curve can be represented by the relation

$$\mathbf{r} = \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}, \tag{3.6}$$

where $\mathbf{r}$ is the position vector of an arbitrary point of the curve, s an arc of the curve, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors of the axes x, y and z, respectively. It follows that

$$\begin{gathered}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{t},\\ \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{1}{\rho}\mathbf{n},\end{gathered} \tag{3.7}$$

where $1/\rho = k_1$ is the first curvature (flexion).

Given that

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \quad \Rightarrow \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \times \mathbf{n} + \mathbf{t} \times \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} \quad \Rightarrow$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{t} \times \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}, \tag{3.8}$$

which follows from

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{1}{\rho}\mathbf{n},$$

vectors $\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}$ and $\mathbf{n}$ are collinear, and their vector product is equal to zero.

Given that $\mathbf{b} \cdot \mathbf{b} = \mathbf{1}$, we obtain

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} \cdot \mathbf{b} = 0. \tag{3.9}$$

From (3.8) and (3.9) it follows that $\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$ (torsion vector) is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$, and thus this vector is collinear with the main normal $\mathbf{n}$, that is,

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -k_2\mathbf{n}. \tag{3.10}$$

Note that $k_2 < 0$, if vectors $\mathbf{k}_2$ and $\mathbf{n}$ have the same direction, when $\mathrm{d}s > 0$, while $k_2 > 0$, if these vectors have opposite directions. The variable k_2 is called the curve torsion(3.6).

Given that $\mathbf{n}\cdot\mathbf{b} = 0$ it follows that

$$\frac{d\mathbf{n}}{ds}\cdot\mathbf{b} + \mathbf{n}\cdot\frac{d\mathbf{b}}{ds} = 0 \quad \Rightarrow$$
$$\frac{d\mathbf{n}}{ds}\cdot\mathbf{b} = -\mathbf{n}\cdot\frac{d\mathbf{b}}{ds} = -\mathbf{n}(-k_2\mathbf{n}) = k_2.$$

By derivation of $\mathbf{n}$, we obtain

$$\mathbf{n} = \mathbf{b}\times\mathbf{t}\,\Big|\,\frac{d}{ds} \quad \Rightarrow$$

$$\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{b}}{ds}\times\mathbf{t} + \mathbf{b}\times\frac{d\mathbf{t}}{ds} =$$
$$-k_2\mathbf{n}\times\mathbf{t} + \mathbf{b}\times\frac{1}{\rho}\mathbf{n} = k_2\mathbf{b} + \frac{1}{\rho}(-\mathbf{t}). \tag{3.11}$$

Relations (3.7), (3.10) and (3.11) represent the so called Frenet-Serret formulas

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \frac{1}{\rho}\mathbf{n}\,(k_1\mathbf{n}),\\ \frac{d\mathbf{n}}{ds} &= -k_1\mathbf{t} + k_2\mathbf{b} = -\frac{1}{\rho}\mathbf{t} + \frac{1}{T}\mathbf{b},\\ \frac{d\mathbf{b}}{ds} &= -k_2\mathbf{n} = -\frac{1}{T}\mathbf{n}. \end{aligned} \tag{3.12}$$

Problem 32

Prove that the curvature radius of a curve defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, is defined by

$$\rho = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-\frac{1}{2}}. \tag{3.13}$$

Proof

Given that the position vector of an arbitrary point on the curve is

$$\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k},$$

it follows that

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx(s)}{ds}\mathbf{i} + \frac{dy(s)}{ds}\mathbf{j} + \frac{dz(s)}{ds}\mathbf{k}$$

and

$$\frac{d\mathbf{t}}{ds} = \frac{d^2x}{ds^2}\mathbf{i} + \frac{d^2y}{ds^2}\mathbf{j} + \frac{d^2z}{ds^2}\mathbf{k}.$$

Given that $\dfrac{d\mathbf{t}}{ds} = k\mathbf{n}$ it follows that

$$k = \left|\frac{d\mathbf{t}}{ds}\right| = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}$$

and we obtain the required result $\rho = \dfrac{1}{k}$.

II

Field theory

III **Solving differential equations**

4. Field theory

Mathematical field theory[1] does not study the physical meaning of a variable defined in a given field. Only the general properties of fields are studied, which are then applied in physics and other areas to specific physical fields. Specific fields are studied in different parts of physics, and in this book some examples will be given for illustration purposes only, in order to help in understanding the theory presented.

4.1 Scalar field

Consider a set D of points in a n–dimensional Euclidean space E^n. If a (real or complex) number y is assigned to each point $M(x_1, x_2, \ldots, x_n) \in D$ by some law, then we say that a scalar (real or complex) function y of n independent variables is defined and we denote it by:

$$y = f(M) \quad \text{or} \quad y = f(x_1, x_2, \ldots, x_n). \tag{4.1}$$

The coordinates $(x_1, x_2, \ldots, x_n)$ of point M are called **independent variables**, and the set D **domain of definition** (or simply domain) of the function f of this point.

However, in physics and some natural sciences, as well as in engineering, the term "field" is used to denote a part of space (area) in which a physical phenomenon is observed ("felt"). The term " **scalar field**", is used here in the mathematical sense, to denote the domain of definition of a scalar function. Thus, hereinafter, the term "field" will be used instead of "domain", and the previous statement can be rephrased as follows: if a function f assigns a scalar (real or complex number) to each point in D then a **scalar field** is defined in D.

Note that the value of the function depends only on the points in space and not on the selected specific coordinate system. Therefore, it should be kept in mind that the value of the function f at any point $M \in D$ does not depend on a selected specific coordinate system. However, its functional form depends on the coordinate system. To emphasize this fact, it is also common to

[1] In mathematics, the term "field" is also used for an algebraic structure consisting of a set and two operations with certain properties, such as, for example, a set of real numbers with operations of addition and multiplication.

denote the function as $f(M)$ rather than $f(x_1,\ldots,x_n)$, as in relation (4.1). When a function is expressed by coordinates, it is said that it is given in analytical form.

Examples of scalar fields include the following: *temperature, mass, mass density, electric charge, pressure*, etc.

Since time t can also be one of the variables, we will call a field that does not (explicitly) depend on time a **stationary field**.

Definition

The geometric location of the points in which the function $f(M) = f(x_1,x_2,x_3)$ has a constant value C:

$$f(M) = f(x_1,x_2,x_3) = C, \tag{4.2}$$

is called an **equiscalar surface** of a scalar field.

Definition

The geometric location of the points in which the function $f(M) = f(x_1,x_2)$ has a constant value C:

$$f(M) = f(x_1,x_2) = C, \tag{4.3}$$

is called an **equiscalar line** of a scalar field.

4.1.1 Directional derivative. Gradient

The application of mathematical analysis to the study of a scalar field $f(M)$ allows for describing its local properties, i.e. changes of $f(M)$ when moving from point M to a close point N.

Consider a scalar field given by the function $f(x,y,z) = f(M)$, within a Cartesian coordinate system. It is known that the first partial derivatives of the scalar function f represent the rates of the change of the function in the directions of the coordinate axes. However, it is natural to extend this concept beyond the three directions (in the 3–D space). Extending this idea to include the change of the function in any possible direction brings about the concept of directional derivative of a (scalar) function.

In order to determine this derivative, let us observe a point M in space and a direction through this point, determined by the unit vector $\mathbf{e}$. Let N be a point at this direction, denoted by ℓ, where the distance between M and N is denoted by Δs (Fig. 4.1)

$$\overrightarrow{MN} = \Delta s\mathbf{e}. \tag{4.4}$$

Let us denote the difference of the function f in points M and N by $\Delta f = f(N) - f(M)$.

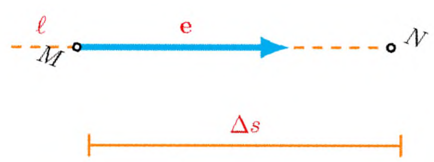


Figure 4.1: Direction of change.

Definition

If the limit value

$$\lim_{\Delta s \to 0} \frac{f(N) - f(M)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}, \tag{4.5}$$

exists, then it is called the **derivative** of the function f, at point M, **in the direction of e**, and denoted by

$$D_{\mathbf{e}} f = \frac{\mathrm{d}f}{\mathrm{d}s}.$$

It is obvious that this derivative represents the rate of change of function f, at point M, i the direction of $\mathbf{e}$. Both ways of denoting, $D_{\mathbf{e}} f$ or $\mathrm{d}f/\mathrm{d}s$, are common, but $D_{\mathbf{e}} f$ is more convenient as it indicates the direction of change.

The directional derivative of a function, as per definition (4.5), does not depend on the choice of the coordinate system. However, in order to calculate specific values of these derivatives, they will be observed with respect to, for example, Cartesian coordinate system, and $f(M)$ will be represented as a function of $f(x,y,z)$. To this end, we will observe the change in the function f while moving from point $M(x,y,z)$ in a given field to a close point $N(x+\Delta x, y+\Delta y, z+\Delta z)$ in the same field in the direction of $\mathbf{e}$. Further, based on the definition of the partial derivative of a scalar function with respect to, for example, x, we obtain:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f_x}{\Delta x}, \tag{4.6}$$

which can then be rewritten, in case of a differentiable function, as

$$\Delta f_x = \frac{\partial f}{\partial x} \cdot \Delta x + \delta_x \cdot \Delta x, \quad \text{gde} \quad \delta_x \to 0 \quad \text{kada} \quad \Delta x \to 0. \tag{4.7}$$

The two remaining increments Δf_y and Δf_z are obtained in the same way, and thus the total increment of the function f can be expressed as

$$\begin{aligned} \Delta f &= f(N) - f(M) = \\ &= \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y + \frac{\partial f}{\partial z} \cdot \Delta z + \delta_x \cdot \Delta x + \delta_y \cdot \Delta y + \delta_z \cdot \Delta z. \end{aligned} \tag{4.8}$$

Further, in Cartesian coordinates

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}. \tag{4.9}$$

Assume now that $\Delta s \to 0$, iff $\Delta x \to 0$, $\Delta y \to 0$ i $\Delta z \to 0$. According to Fig. 4.2,

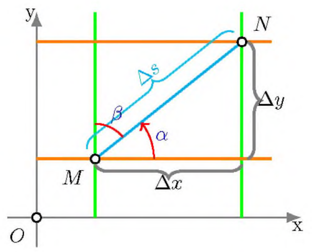


Figure 4.2: Increments in the directions of x and y axes.

it is obvious that

$$\frac{\Delta x}{\Delta s} = \cos\alpha, \quad \frac{\Delta y}{\Delta s} = \cos\beta, \quad \text{and by analogy} \quad \frac{\Delta z}{\Delta s} = \cos\gamma, \tag{4.10}$$

where α, β and γ are angles between the vector $\overrightarrow{MN}$ and positive directions of axes x, y and z, respectively.

Thus, the derivative of a scalar function f in the direction of $\mathbf{e}$, can also be represented as follows

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}s} &= \lim_{\Delta s\to 0}\frac{\Delta f}{\Delta s} = \lim_{\Delta s\to 0}\frac{\frac{\partial f}{\partial x}\cdot\Delta x+\frac{\partial f}{\partial y}\cdot\Delta y+\frac{\partial f}{\partial z}\cdot\Delta z+\delta_x\cdot\Delta x+\delta_y\cdot\Delta y+\delta_z\cdot\Delta z}{\Delta s}\\ &= \frac{\partial f}{\partial x}\cdot\cos\alpha+\frac{\partial f}{\partial y}\cdot\cos\beta+\frac{\partial f}{\partial z}\cdot\cos\gamma+\\ &+\cos\alpha\lim_{\Delta s\to 0}\delta_x+\cos\beta\lim_{\Delta s\to 0}\delta_y+\cos\gamma\lim_{\Delta s\to 0}\delta_z=\\ &= \frac{\partial f}{\partial x}\cdot\cos\alpha+\frac{\partial f}{\partial y}\cdot\cos\beta+\frac{\partial f}{\partial z}\cdot\cos\gamma.\end{aligned} \tag{4.11}$$

The directional derivative can be obtained in the following manner as well. Observe the function $f(x,y,z)$ defined in the neighborhood of point $M(a,b,c)$, which lies on direction ℓ. Let this direction be determined by the unit vector $\mathbf{e}$, and let $\mathbf{r}_M$ and $\mathbf{r}$ be the position vectors of points M and $N\in\ell$, respectively, (Fig. 4.3)

$$\Delta s=\overline{MN},\quad \mathbf{r}=\mathbf{r}_M+\Delta s\cdot\mathbf{e},$$

then

$$x=a+\Delta s\cdot\cos\alpha,\quad y=b+\Delta s\cdot\cos\beta,\quad z=c+\Delta s\cdot\cos\gamma. \tag{4.12}$$

We further obtain the following for the directional derivative

$$\begin{aligned}D_{\mathbf{e}}f=\frac{\mathrm{d}f}{\mathrm{d}s} &= \lim_{N\to M}\frac{f(N)-f(M)}{\Delta s}=\\ &= \lim_{\Delta s\to 0}\frac{f(\mathbf{r})-f(\mathbf{r}_M)}{\Delta s}=\\ &= \lim_{\Delta s\to 0}\frac{f(\mathbf{r}_M+\Delta s\mathbf{e})-f(\mathbf{r}_M)}{\Delta s}.\end{aligned}$$

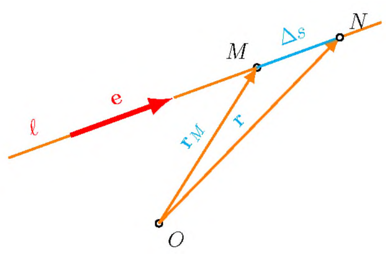


Figure 4.3: Increment Δs.

Converting the function $f(N)$ into a power series in the neighbourhood of point $M(a,b,c)$ and using (4.12), we obtain

$$\begin{aligned}f(N) &= f(x,y,z)=f(a+\Delta s\cdot\cos\alpha,\,b+\Delta s\cdot\cos\beta,\,c+\Delta s\cdot\cos\gamma)= \\ &= f(M)+\frac{1}{1!}\left(\left.\frac{\partial f}{\partial x}\right|_M\cdot\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_M\cdot\cos\beta+\left.\frac{\partial f}{\partial z}\right|_M\cdot\cos\gamma\right)\cdot\Delta s+\delta(N)\cdot\Delta s,\end{aligned} \tag{4.13}$$

where $\lim_{N\to M}\delta(N)=0$, and it thus follows that

$$\begin{aligned}&\frac{f(N)-f(M)}{\Delta s}=\\ &\qquad=\left.\frac{\partial f}{\partial x}\right|_M\cdot\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_M\cdot\cos\beta+\left.\frac{\partial f}{\partial z}\right|_M\cdot\cos\gamma+\delta(N),\end{aligned} \tag{4.14}$$

that is

$$\begin{aligned}D_{\mathbf{e}}f=\frac{\mathrm{d}f}{\mathrm{d}s} &= \lim_{\Delta s\to 0}\frac{f(N)-f(M)}{\Delta s}=\left.\frac{\partial f}{\partial x}\right|_M\cdot\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_M\cdot\cos\beta+\left.\frac{\partial f}{\partial z}\right|_M\cdot\cos\gamma=\\ &= f_x\cdot\cos\alpha+f_y\cdot\cos\beta+f_z\cdot\cos\gamma.\end{aligned} \tag{4.15}$$

Thus, we obtain the same result as in relation (4.11).

As an infinite number of directions pass through each point, an infinite number of derivatives can be obtained in those directions. However, if we observe a coordinate system, for example, Cartesian, each of these derivatives can be expressed by a first partial derivative of the function f at point M as follows. Let point M be determined by the position vector $\mathbf{r}_M$ and let $\mathbf{e}$ be a unit vector. Let now ℓ be the line through point M, which can be represented as follows

$$\mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j} + z(s)\,\mathbf{k} = \mathbf{r}_M + s\mathbf{e} \quad (s \geq 0,\ |\mathbf{e}| = 1), \tag{4.16}$$

where $\mathbf{r}(s)$ is the position vector, depending on parameter s (arc length).

Observe the derivative of the function f along the curved line ℓ, then $D_{\mathbf{e}} f = \dfrac{\mathrm{d}f}{\mathrm{d}s}$ is the derivative of the function $f[x(s), y(s), z(s)]$, which depends on the length of the arc s.

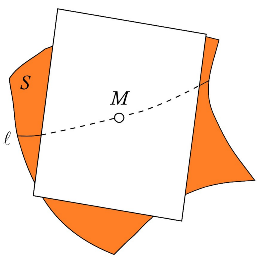


Figure 4.4: Line ℓ on surface S and the tangent plane at point M.

Thus, assuming that f has continuous partial derivatives, and applying the rule on derivation of complex functions, we obtain

$$\begin{aligned} D_{\mathbf{e}} f = \frac{\mathrm{d}f}{\mathrm{d}s} &= \frac{\partial f}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial f}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}s} = \\ &= \frac{\partial f}{\partial x}\cdot x' + \frac{\partial f}{\partial x}\cdot y' + \frac{\partial f}{\partial x}\cdot z', \end{aligned} \tag{4.17}$$

where ($'$) denotes the derivative with respect to the parameter s.

Differentiating the vector function $\mathbf{r}(s)$, we obtain from (4.16) [2] :

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{t} = x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k} = \mathbf{e}. \tag{4.18}$$

Thus, in this case $\mathbf{e}$ has the direction of the tangent.

Bearing in mind the scalar product and the relation (4.18), the expression (4.17) can be transformed as follows

$$\begin{aligned} D_{\mathbf{e}} f &= \left(\frac{\partial f}{\partial x}\cdot\mathbf{i} + \frac{\partial f}{\partial y}\cdot\mathbf{j} + \frac{\partial f}{\partial z}\cdot\mathbf{k}\right)\left(x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k}\right) = \\ &= \left(\frac{\partial f}{\partial x}\cdot\mathbf{i} + \frac{\partial f}{\partial y}\cdot\mathbf{j} + \frac{\partial f}{\partial z}\cdot\mathbf{k}\right)\mathbf{e}, \end{aligned}$$

which leads us to the introduction of the following vector.

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[2] Let the curved line ℓ be given in the parametric form

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k},$$

where the arc of the curve s is the parameter. Then

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{j} + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{k} = \mathbf{t},$$

where $\mathbf{t}$ is the directional vector of unit magnitude of the tangent at a point of the curved line ℓ, as

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right| = |\mathbf{t}| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2} = \sqrt{\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\mathrm{d}s^2}} = 1.$$

Definition

A vector determined by the relation

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{4.19}$$

is called the **gradient** of the scalar function f.

As the gradient of a scalar function is a vector value, it follows for this vector that

$$\text{magnitude: } |\operatorname{grad} f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}, \tag{4.20}$$

$$\text{direction: } \cos\alpha = \frac{\partial f/\partial x}{|\operatorname{grad} f|}, \cos\beta = \frac{\partial f/\partial y}{|\operatorname{grad} f|}, \cos\gamma = \frac{\partial f/\partial z}{|\operatorname{grad} f|}. \tag{4.21}$$

The directional derivative can now be represented as

$$D_{\mathbf{e}} f = \frac{\mathrm{d}f}{\mathrm{d}s} = \mathbf{e} \cdot \operatorname{grad} f. \tag{4.22}$$

The geometric interpretation of this product is the projection of the gradient on the direction determined by vector **e**, that is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \operatorname{proj}_{\mathbf{e}} \operatorname{grad} f = |\operatorname{grad} f| \cdot \cos\varphi, \tag{4.23}$$

where φ is the angle between grad$f$ and **e**.

It follows from this definition that $\mathrm{d}f/\mathrm{d}s$ has maximum value when $\cos\varphi = 1 \Rightarrow \varphi = 0$. Thus, the scalar field changes most rapidly in the direction of the gradient, i.e. the gradient determines the direction in which the scalar field changes most rapidly.

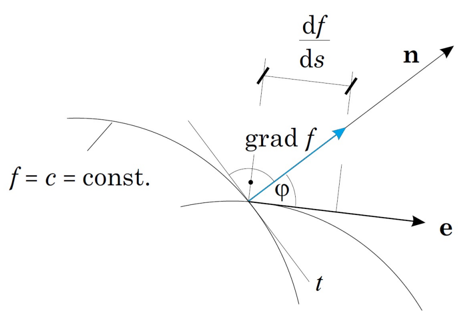


Figure 4.5: Gradient of function f.

In the special case, when the derivative is obtained in the direction of the $+Ox$ axis, then **e=i**, and we obtain

$$D_{\mathbf{i}} f = \mathbf{i} \cdot \operatorname{grad} f = \mathbf{i} \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{\partial f}{\partial x}\mathbf{i} \cdot \mathbf{i} = \frac{\partial f}{\partial x}. \tag{4.24}$$

Theorem 5

Let $f(M) = f(x,y,z)$ be a scalar function, the first partial derivatives of which are continuous functions. Then a vectorgrad f exists, whose magnitude and direction are independent of the coordinate system. If grad f at point M is not equal to zero, then it has the direction of maximum increase of function f at point M.

We will prove this theorem later.

Theorem 6

Let the gradient of the function $u = f(x,y,z)$ at point M be different from zero. It is then orhtogonal[3]to each line ℓ, which passes through M, and lies in the equiscalar surface $f = const$.

Proof

Observe line ℓ, passing through point M and lying in surface $f = const$. (Fig. 4.6). As the value of the function does not change when the point moves along the line ℓ (because it lies in $f = const$.), then

$$\frac{df}{ds} = 0.$$

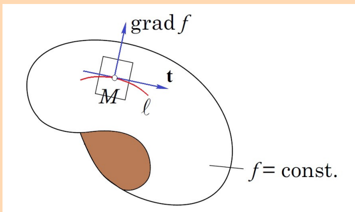


Figure 4.6: Gradient of function f.

As, on the other hand, the derivative of the function along the line ℓ is (4.23)

$$D_{\mathbf{e}}f = \frac{df}{ds} = \mathbf{e}\cdot \operatorname{grad} f = |\operatorname{grad} f|\cdot \cos(\operatorname{grad} f, \mathbf{e}) = 0,$$

then, assuming that grad$f \neq 0$, we obtain $\cos\varphi = 0$ (Fig. 4.5). Thus, if $\mathbf{e}$ is the unit vector of the tangent line ℓ, it follows that the gradient is orthogonal to the equiscalar surface.

The **derivative of the function f in the direction of the tangent $\mathbf{t}$** ($\mathbf{t}$ is the unit vector of the tangent to the curved line C at point M) is, according to (4.23)

$$D_{\mathbf{t}}f = \frac{df}{ds} = \mathbf{t}\cdot \operatorname{grad} f = \tag{4.25}$$
$$\left(\frac{dx}{ds}\cdot\mathbf{i} + \frac{dy}{ds}\cdot\mathbf{j} + \frac{dz}{ds}\cdot\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\cdot\mathbf{i} + \frac{\partial f}{\partial y}\cdot\mathbf{j} + \frac{\partial f}{\partial z}\cdot\mathbf{k}\right) = \frac{d\mathbf{r}}{ds}\cdot \operatorname{grad} f.$$

From here we obtain

$$df = d\mathbf{r}\cdot \operatorname{grad} f. \tag{4.26}$$

[3]By "orthogonal to the line at point M" it is assumed that it is orthogonal to the tangent plane, passing through M.

Thus, when a scalar function is differentiable, then its total differential is equal to the scalar product of the gradient of the function and the position vector differential. This indicates one way to calculate the gradient of a function. Namely, when a function differential can be represented as a scalar product of two vectors, one of which is d**r**, then the other factor is equal to the gradient of the function.

4.1.2 Partial gradient of a scalar function

Observe a scalar function f depending on two vectors **u** and **v**:

$$f = f(\mathbf{u}, \mathbf{v}), \tag{4.27}$$

where these two vectors can be represented, with respect to the Cartesian coordinate system, as follows

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3,$$
$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$

The field of this function is determined by points U and V, which represent the end points of vectors **u** and **v**. We used here the following symbols: $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$.

Let us assume that the vector **v** is constant and determine the gradient of the function f under this condition. According to (4.19) we obtain

$$\mathrm{grad}_{\mathbf{u}}\, f(\mathbf{u}, \mathbf{v}) = \frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{\partial f}{\partial u_3}\mathbf{e}_3, \tag{4.28}$$

which is called the **partial gradient** of the scalar function f, along the vector **u**.

Similarly, we define the partial gradient along the other vector. We can also extend this definition to an arbitrary, but finite, number of vectors. In this case all vectors except one would be considered constant.

4.1.3 Properties of gradient

a) grad $C = 0\,(C = const.)$,
b) grad $(U+V)$=grad U+grad V, gde je $U = U(M), V = V(M)$,
c) grad $(U\cdot V)$=$V\cdot$grad U+$U\cdot$ grad V,
d) grad (CU)=C grad U, C=const.,
e) grad (U/V)=$(1/V^2)(V\cdot$grad U-$U\cdot$ grad $V)$,
f) grad $f(U) = f'_U\cdot$grad U.

Based on these properties, the following is true

$$\mathrm{grad}\,(C_1U_1 + C_2U_2) = C_1\mathrm{grad}\,U_1 + C_2\mathrm{grad}\,U_2. \tag{4.29}$$

Operators having this property are called **linear operators**. Thus, the gradient is a linear operator.

Proof

a)

$$\mathrm{grad}C = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)C = \frac{\partial C}{\partial x}\mathbf{i} + \frac{\partial C}{\partial y}\mathbf{j} + \frac{\partial C}{\partial z}\mathbf{k} = 0+0+0 = 0.$$

b)

$$\begin{aligned}
\mathrm{grad}(U+V) &= \\
&= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(U+V) = \\
&= \frac{\partial(U+V)}{\partial x}\mathbf{i}+\frac{\partial(U+V)}{\partial y}\mathbf{j}+\frac{\partial(U+V)}{\partial z}\mathbf{k} = \\
&= \frac{\partial U}{\partial x}\mathbf{i}+\frac{\partial U}{\partial y}\mathbf{j}+\frac{\partial U}{\partial z}\mathbf{k}+\frac{\partial V}{\partial x}\mathbf{i}+\frac{\partial V}{\partial y}\mathbf{j}+\frac{\partial V}{\partial z}\mathbf{k} = \\
&= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)U+\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)V = \\
&= \mathrm{grad}U+\mathrm{grad}V.
\end{aligned}$$

c)

$$\begin{aligned}
\mathrm{grad}(U\cdot V) &= \\
&= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(U\cdot V) = \\
&= \frac{\partial(U\cdot V)}{\partial x}\mathbf{i}+\frac{\partial(U\cdot V)}{\partial y}\mathbf{j}+\frac{\partial(U\cdot V)}{\partial z}\mathbf{k} = \\
&= \frac{\partial U}{\partial x}\mathbf{i}\cdot V+U\cdot\frac{\partial V}{\partial x}\mathbf{i}+\frac{\partial U}{\partial y}\mathbf{j}\cdot V+U\cdot\frac{\partial V}{\partial y}\mathbf{j}+\frac{\partial U}{\partial z}\mathbf{k}\cdot V+U\cdot\frac{\partial V}{\partial z}\mathbf{k} = \\
&= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)U\cdot V+U\cdot\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)V = \\
&= \mathrm{grad}U\cdot V+U\cdot\mathrm{grad}V.
\end{aligned}$$

As an exercise, prove that the remaining three gradient properties are also valid.

4.1.4 Nabla operator or Hamilton operator

Introducing the differential operator, called "nabla"[4] or Hamiltonian[5], defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}, \tag{4.30}$$

the gradient of the function f can be expressed in the following form

$$\mathrm{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}. \tag{4.31}$$

The symbol ∇f is very often used in technics to denote the gradient.

Properties of the nabla operator ∇

Let f and g be two scalar functions, and **a** and **b** two vector functions. The properties of the nabla operator can then be expressed as follows

[4] According to the Hebrew letter ∇ used to denote this operator.
[5] William Rowan Hamilton (1805-1865), Irish mathematician, known for his work in dynamics.

a) $\nabla(f \cdot g) = g\nabla f + f\nabla g$,
b) $\nabla \cdot (f\mathbf{a}) = (\nabla f) \cdot \mathbf{a} + f(\nabla \cdot \mathbf{a})$,
c) $\nabla \times (f\mathbf{a}) = (\nabla f) \times \mathbf{a} + f(\nabla \times \mathbf{a})$,
d) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$,
e) $\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$.[6]
f) $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b})$.

4.1.5 Laplace or delta operator

Let us now define an operator, of scalar nature, as follows

$$\begin{aligned}\nabla \cdot \nabla = \nabla^2 &= \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) = \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta.\end{aligned} \tag{4.32}$$

The symbol Δ - "delta", stands for the **Laplace operator**[7] or Laplacian.

[6] Note that $\mathbf{a} \cdot \nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$

[7] Pierre Simon Marquis De Laplace (1749-1827), great French mathematician. He laid the foundations of potential theory and made major contributions in mechanics, astronomy, as well as in the field of special functions and probability theory. It is interesting to note that Napoleon Bonaparte was also his student for one year.

4.2 Vector field

4.2.1 Vector function. Vector field

Let us assign to each point M, of a region $\mathscr{D}$, according to some law, a value of a vector $\mathbf{v}$. We then say that a vector function $\mathbf{v}$=$\mathbf{v}(M)$ is defined. The set $\mathscr{D}$, to which values of the argument belong, is called the domain or the **vector field** of this function. Thus the vector field is the domain of a vector function.

However, as each point M is determined by a position vector $\mathbf{r}$, the previous definition can be reformulated as follows

> if to each value of position vector $\mathbf{r}\in\mathbb{V}$, where $\mathbb{V}$ is the three-dimensional vector space with base vectors $\mathbf{e}_i$ $(i = 1,2,3)$, a value of a vector $\mathbf{v}$ is assigned according to some law, we say that a vector function of the argument $\mathbf{r}$ is defined

$$\mathbf{v} = \mathbf{v}(\mathbf{r}) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \tag{4.33}$$

We have expressed here the position vector by its components, in relation to a coordinate system, and consequently also the function itself. As in the Cartesian coordinate system

$$v_1 = v_1(x,y,z) \quad v_2 = v_2(x,y,z) \quad v_3 = v_3(x,y,z) \tag{4.34}$$

it follows that each of these relations defines a scalar field. Thus, the study of vector fields comes down to the study of three (if observed in a three-dimensional space) scalar fields.

Examples include: *the gravitational field of the Earth, the velocity field of a moving fluid, the gradient of a scalar function* etc. As an example, we can take any analytically given vector function, such as $\mathbf{v}= xy\mathbf{i}-2yz\mathbf{j}+x\mathbf{k}$, whose domain is a vector field.

Some typical examples of vector fields are represented in figures 4.7 a,b,c,d.

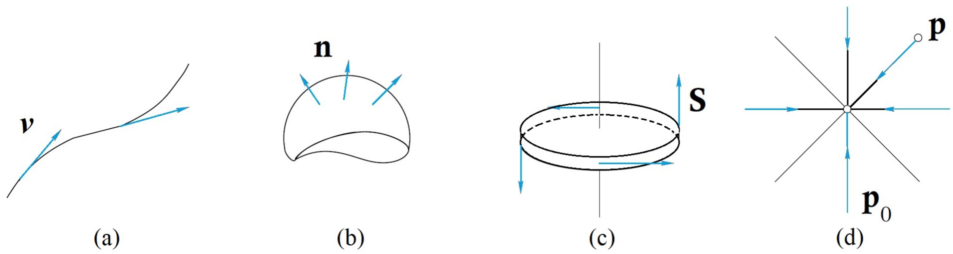


Figure 4.7: Examples of vector fields.

If the vector field is independent of time, it is called a stationary vector field.

Definition

A **vector line** (line of forces) ℓ is the geometric place of the points of a vector field in which the vector function $\mathbf{v}(M)$ has the direction of the tangent to this line at given points, i.e. a line where at each point the direction of the vector coincides with the direction of the tangent of the curve at that point.

(a) Unit vector of tangent **t** to vector line ℓ. (b) Increment **dr** of vector line ℓ.

Figure 4.8: Vector line $\mathbf{v}(\mathbf{r})$.

As the vector of the tangent **t**, namely **dr**, and vector **v** are collinear, it follows from the definition of a vector line that it's equation is

$$\mathbf{dr}\times\mathbf{v}=0, \tag{4.35}$$

or, on the basis of (1.28), as follows (see Fig. 4.8):

$$\mathbf{dr}=\mathbf{v}\cdot \mathrm{d}t,\quad \text{where } t \text{ is a parameter.} \tag{4.36}$$

In the Cartesian coordinate system (4.36) becomes

$$\mathrm{d}x\mathbf{i}+\mathrm{d}y\mathbf{j}+\mathrm{d}z\mathbf{k}=(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\cdot \mathrm{d}t. \tag{4.37}$$

As **i**, **j** and **k** are linearly independent, from the last relation we finally obtain

$$\frac{\mathrm{d}x}{v_1}=\frac{\mathrm{d}y}{v_2}=\frac{\mathrm{d}z}{v_3}=\mathrm{d}t. \tag{4.38}$$

This relation represents the differential equation of the vector line.

The directional derivative of a vector function is defined similarly to the derivative of a scalar function.

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}=\frac{\mathrm{d}v_1}{\mathrm{d}s}\mathbf{i}+\frac{\mathrm{d}v_2}{\mathrm{d}s}\mathbf{j}+\frac{\mathrm{d}v_3}{\mathrm{d}s}\mathbf{k}, \tag{4.39}$$

where

$$\begin{aligned}\frac{\mathrm{d}v_1}{\mathrm{d}s}&=\frac{\partial v_1}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial v_1}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial v_1}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}s},\\ \frac{\mathrm{d}v_2}{\mathrm{d}s}&=\frac{\partial v_2}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial v_2}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial v_2}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}s},\\ \frac{\mathrm{d}v_3}{\mathrm{d}s}&=\frac{\partial v_3}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial v_3}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial v_3}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}s}.\end{aligned} \tag{4.40}$$

Observe a closed oriented curve in the vector field **v**, which we will call a contour. A vector line (l) passes through each point of this contour.

Definition

The geometric locations of vector lines that pass through the points of a contour in a vector field **v**, form a surface called the **solenoid** (**tube** or **vector surface** or **vector tube**), see Fig. 4.9.

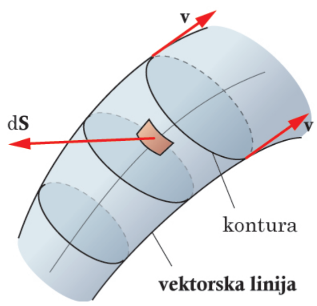


Figure 4.9: Vector surface.

In order to obtain the equation of this surface, let us denote by S the area of the side of the tube, and by $\mathrm{d}\mathbf{S}$ the vector element of this surface. According to the definition of the solenoid[8] it follows that the vector surface element is orthogonal to the vector $\mathbf{v}$, and thus

$$\mathbf{v}\cdot \mathrm{d}\mathbf{S} = 0. \tag{4.41}$$

Consequently, if S is the total area of the side, then

$$\int_S \mathbf{v}\cdot \mathrm{d}\mathbf{S} = 0. \tag{4.42}$$

4.2.2 Divergence and rotor

Observe a differentiable vector function $\mathbf{v}$, which can be represented, with respect to the Cartesian coordinate system, in the following form

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}. \tag{4.43}$$

Definition

A scalar function, defined by the relation

$$\operatorname{div}\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{4.44}$$

is called the **divergence** of the vector function $\mathbf{v}$ or divergence of the vector field defined by $\mathbf{v}$.

A more suitable form for denoting divergence is by way of the aforementioned ∇ operator, as follows

$$\begin{aligned}\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) = \\ &= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right).\end{aligned} \tag{4.45}$$

Theorem 7

The value $\operatorname{div}\mathbf{v}$ depends only of the points in space (and naturally of the value of the function $\mathbf{v}$), but not on the choice of the coordinate system.

R Note that it is possible to define divergence in such a way that it is obvious that it does not depend on the choice of the coordinate system, which will be done later.

[8] Solenoid - form Greek word $\sigma\omega\lambda\eta\nu$ - tube.

Definition

A vector function, defined by

$$\operatorname{rot}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \tag{4.46}$$

$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

is called the **rotor** of the vector function **v** or the rotor of the vector field, defined by the function **v**.

Theorem 8

The magnitude and direction of the vector $\operatorname{rot}\mathbf{v}$ do not depend on the specifically selected Cartesian coordinate system.

This statement will be proved later.

4.2.3 Classification of vector fields

Definition

A vector field, in which for all its points the following is true

$$\operatorname{rot}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{v} \neq 0, \tag{4.47}$$

is called a **potential** or irrotational or lamellar field.

Definition

A vector field, in which for all its points the following is true

$$\operatorname{rot}\mathbf{v} \neq 0, \quad \operatorname{div}\mathbf{v} = 0, \tag{4.48}$$

is called a solenoidal or **rotational** field.

Definition

A vector field, in which for all its points the following is true

$$\operatorname{rot}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{v} = 0, \tag{4.49}$$

is called a **Laplace field**.

Definition

A vector field, in which for all its points the following is true

$$\operatorname{rot}\mathbf{v} \neq 0, \quad \operatorname{div}\mathbf{v} \neq 0, \tag{4.50}$$

is called a **complex field**.

R Note that the study of a complex field can be reduced to one potential and one solenoidal field.

Let us express a complex field in the form

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \tag{4.51}$$

so that

$$\operatorname{rot}\mathbf{v}_1 = 0, \quad \operatorname{div}\mathbf{v}_1 \neq 0, \tag{4.52}$$

$$\operatorname{rot}\mathbf{v}_2 \neq 0, \quad \operatorname{div}\mathbf{v}_2 = 0. \tag{4.53}$$

Then, as

$$\operatorname{div}\mathbf{v} = \operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2) = \operatorname{div}\mathbf{v}_1 + \operatorname{div}\mathbf{v}_2 = \operatorname{div}\mathbf{v}_1 \neq 0, \tag{4.54}$$

$$\operatorname{rot}\mathbf{v} = \operatorname{rot}(\mathbf{v}_1 + \mathbf{v}_2) = \operatorname{rot}\mathbf{v}_1 + \operatorname{rot}\mathbf{v}_2 = \operatorname{rot}\mathbf{v}_2 \neq 0. \tag{4.55}$$

the previous statement is proved.

4.2.4 Potential

Assume that the vector function $\mathbf{v}$ can be represented as the gradient of a scalar position function $\varphi(\mathbf{r})$, i.e.

$$\mathbf{v} = \operatorname{grad}\varphi. \tag{4.56}$$

The scalar function φ defined in this way is called the **potential** of the vector field $\mathbf{v}$.

Theorem 9

The field of the vector function **v=grad** φ is a potential field.

Proof

As, according to the assumption

$$\mathbf{v} = \operatorname{grad}\varphi \quad \Rightarrow \quad v_x = \frac{\partial\varphi}{\partial x}, \quad v_y = \frac{\partial\varphi}{\partial y}, \quad v_z = \frac{\partial\varphi}{\partial z}, \tag{4.57}$$

it follows that

$$\frac{\partial v_z}{\partial y} = \frac{\partial^2\varphi}{\partial z\partial y}, \quad \frac{\partial v_y}{\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}, \quad \Rightarrow \quad \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0. \tag{4.58}$$

Similarly, we obtain

$$\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0 \quad \text{i} \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0. \tag{4.59}$$

On basis of these relations we obtain $\text{rot}\,\mathbf{v} = 0$, given that

$$\begin{aligned}\text{rot}\,\mathbf{v} &= \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k} = \\ &= 0. \end{aligned} \tag{4.60}$$

As, in general, the following is true

$$\begin{aligned}\text{div}\,\mathbf{v} &= \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \\ &\neq 0 \end{aligned} \tag{4.61}$$

the theorem is proven.

4.2.5 Examples of potential

Potential of a position vector

Observe the vector field $\mathbf{v} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

As $\dfrac{\partial v_x}{\partial x} = \dfrac{\partial v_y}{\partial y} = \dfrac{\partial v_z}{\partial z} = 1$ it follows that $\text{div}\mathbf{v} = 3 \neq 0$.

It further follows that

$$\text{rot}\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k} = 0, \tag{4.62}$$

so we can conclude that the vector field $\mathbf{v} = \mathbf{r}$ is potential.

Potential forces

If there exists a scalar function U such that the force $\mathbf{S}$ can be represented in the form

$$\mathbf{S} = \text{grad}\,U, \tag{4.63}$$

then it is said that the force is **conservative** and that there exists a **potential of the force** U. Observe, for example, the gravitational force

$$\mathbf{S} = -\gamma \frac{m \cdot m_0}{R^2}\mathbf{r}, \quad \mathbf{r} = \frac{\mathbf{R}}{|\mathbf{R}|}, \tag{4.64}$$

where m and m_0 are masses that are mutually attracted, γ is the gravitational constant, $\mathbf{R}$ the position vector of one material point with respect to another, and R the magnitude of the position vector.

The potential of this force is given by the expression (see (4.98))

$$U = \gamma \frac{m_0}{R}. \tag{4.65}$$

Stationary electrostatic field

In electrodynamics, the problem of determining the strength of electric and magnetic fields can be reduced to determining their potential. Let's start with Maxwell's[9] equations for electromagnetic field in vacuum:

[9] Maxwell James Clark (1831-1879), British physicist. He researched in many areas of physics, and his most significant works are related to electromagnetic phenomena. He formulated four equations laying out the principle

$$\text{div}\mathbf{E} = \frac{1}{\varepsilon_o}\rho, \quad \text{div}\mathbf{B} = 0, \quad \text{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{rot}\mathbf{B} = \mu_o\mathbf{j} + \varepsilon_o\mu_o\frac{\partial \mathbf{E}}{\partial t},$$

where $\mathbf{E}(x,y,z,t)$ and $\mathbf{B}(x,y,z,t)$ are the strength of the electric field and the induction of the magnetic field, respectively, ε_0 is the dielectric constant in the vacuum, μ_0 is the magnetic permeability of the vacuum, and $\rho(x,y,z,t)$ and $\mathbf{j}(x,y,z,t)$ are the charge density and current density, respectively.

Observe the second and third equations (which are called sourceless equations in the literature, because they do not include the charge density and current density, which characterize the field sources). Since the divergence of the rotor, of any vector, is identically equal to zero (4.71), we can write that

$$\mathbf{B} = \text{rot}\mathbf{A} \quad \text{div}\mathbf{B} = \text{div}(\text{rot}\mathbf{A}) = 0,$$

where $\mathbf{A} = \mathbf{A}(x,y,z,t)$. Replacing this in the third Maxwell's equation, we obtain

$$\text{rot}\mathbf{E} = -\frac{\partial}{\partial t}\text{rot}\mathbf{A} = \text{rot}\left(-\frac{\partial \mathbf{A}}{\partial t}\right). \tag{4.66}$$

As the rotor of the gradient of any scalar function id identically equal to zero (4.69), the values $\mathbf{E}$ and $\frac{\partial \mathbf{A}}{\partial t}$ can differ for the gradient of a scalar function Φ, where $\Phi = \Phi(x,y,z,t)$. Thus

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\Phi. \tag{4.67}$$

The vector function $\mathbf{A}(x,y,z,t)$ and scalar function $\Phi(x,y,z,t)$ are called **vector** and **scalar potential**, respectively.

In order to identify the physical meaning of the scalar potential, let us assume that the electromagnetic field is stationary, i.e. that it does not to change over time. Then $\frac{\partial \mathbf{A}}{\partial t}$=0, and thus

$$\mathbf{E} = -\text{grad}\Phi. \tag{4.68}$$

By scalar multiplication by the displacement vector $\mathbf{r}$ we obtain

$$\mathbf{E}\cdot \text{d}\mathbf{r} = -\text{grad}\Phi\cdot\text{d}\mathbf{r} = -\frac{\partial\Phi}{\partial x}\text{d}x - \frac{\partial\Phi}{\partial y}\text{d}y - \frac{\partial\Phi}{\partial z}\text{d}z = -\text{d}\Phi,$$

and then by integrating along some path from infinity to the point of space in which we observe the field, we obtain

$$\Phi(x,y,z) = -\int_{\infty}^{(x,y,z)} \mathbf{E}\cdot\text{d}\mathbf{r}.$$

Thus, for a stationary electromagnetic field, the scalar potential represents the work that some external force needs to perform against the electric field in order to bring the unit charge of the same sign as the field source from infinity to the observed point (x,y,z). The value of the scalar potential at infinity is assumed to be zero.

Obviously, in the case of a time-varying field, this conclusion is no longer valid.

The vector potential $\mathbf{A}(x,y,z)$ itself has no direct physical interpretation, as opposed to its line integral along some closed contour L. Namely, as

$$\int_L \mathbf{A}\cdot\text{d}\mathbf{l} = \iint_S \text{rot}\mathbf{A}\cdot\text{d}\mathbf{S} = \iint_S \mathbf{B}\cdot\text{d}\mathbf{S},$$

according to which changes in the electric field cause changes in the magnetic field and vice versa. He formulated the law of distribution of the velocity of molecules in a gas. He is considered one of the founders of the kinetic theory of gases, along with L. Boltzmann and R. Clausius.

it follows that the movement of the vector potential vector, along any closed contour, is equal to the magnetic flux through any surface bordered by that contour, which is valid in the general case.

Gauge or gradient invariance of electromagnetic field

Note that the functions of scalar and vector potential for a given electromagnetic field are not unambiguous. This is a consequence of the fact that they appear only in the form of their derivatives, and are thus determined only with the accuracy to the terms that are shortened in operations within the specified formulas.

For practice, show that Maxwell's equations do not change (they are invariant) if $\mathbf{A}$ and Φ are changed as follows

$$\Phi_o = \Phi - \frac{\partial f}{\partial t}; \quad \mathbf{A}_o = \mathbf{A} + \mathrm{grad} f,$$

where $f = f(x,y,z,t)$ is a function of the variables x,y,z,t.

Since Maxwell's equations determine the values of $\mathbf{E}$ i $\mathbf{B}$, it follows that an entire family of vector and scalar potentials that satisfy equations (4.66) and (4.68). can be defined for an electromagnetic field. The simplest physical explanation (simplified for the stationary case) is the example of the electrostatic field where the gradient invariance allows us to choose the reference level (the level at which the potential energy is equal to zero), in relation to which the potential energy and the potential are calculated. This means that in the definition of potential it is not necessary to assume that the test charge goes from infinity, but rather from some point in space, which thus becomes the reference (zero) level. Regardless of how the reference level is defined, the strength of the electrostatic field remains unchanged.

$$\mathbf{E} = \mathrm{grad}(\Phi + \phi) = \mathrm{grad}\Phi \quad \phi = const.$$

ϕ is here practically the potential of the chosen reference level with respect to infinity.

Electromagnetic potential equations

Let us observe what is obtained when the scalar and vector potentials of the electromagnetic field are inserted in Maxwell's equations for vacuum, where the field sources (current density $\mathbf{j}$ and charge density ρ) appear.

$$\mathrm{rot}\mathbf{B} = \mu_o \mathbf{j} + \varepsilon_o \mu_o \frac{\partial \mathbf{E}}{\partial t},$$
$$\mathrm{rotrot}\mathbf{A} = \mu_o \mathbf{j} + \frac{\partial}{\partial t}\left(-\mathrm{grad}\Phi - \frac{\partial \mathbf{A}}{\partial t}\right)\varepsilon_o \mu_o,$$
$$-\Delta\mathbf{A} + \mathrm{graddiv}\mathbf{A} = \mu_o \mathbf{j} - \left(\mathrm{grad}\frac{\partial \Phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right)\varepsilon_o \mu_o,$$
$$\Delta\mathbf{A} - \varepsilon_o \mu_o \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{j} + \mathrm{grad}\left(\mathrm{div}\mathbf{A} + \varepsilon_o \mu_o \frac{\partial \Phi}{\partial t}\right).$$

Given the gradient invariance of the potential, $\mathbf{A}(x,y,z,t)$ and $\Phi(x,y,z,t)$ can be chosen so that they satisfy the expression

$$\mathrm{div}\mathbf{A} + \varepsilon_o \mu_o \frac{\partial \Phi}{\partial t} = 0,$$

and it follows that

$$\Delta\mathbf{A} - \varepsilon_o \mu_o \frac{\partial^2 \mathbf{A}}{\partial \mathbf{t^2}} = -\mu_o \mathbf{j}.$$

On the other hand

$$\operatorname{div}\mathbf{E} = \frac{1}{\varepsilon_o}\rho,$$

$$\operatorname{div}\left(-\operatorname{grad}\Phi - \frac{\partial \mathbf{A}}{\partial t}\right) = \frac{1}{\varepsilon_o}\rho,$$

$$\Delta\Phi + \frac{\partial}{\partial t}(\operatorname{div}\mathbf{A}) = -\frac{1}{\varepsilon_o}\rho.$$

Given that $\operatorname{div}\mathbf{A} = -\varepsilon_o\mu_o\dfrac{\partial\Phi}{\partial t}$ it follows that

$$\Delta\Phi - \varepsilon_o\mu_o\frac{\partial^2\Phi}{\partial t^2} = -\frac{1}{\varepsilon_o}\rho.$$

Thus, instead of four Maxwell partial differential equations that are coupled, in which **E** and **B** are unknown, we obtain four uncoupled equations, which are easier to solve, and in which the variables **A** and Φ are unknown.

Properties of divergence

a) div(c**a**)=c·div **a**, c=const.
b) div(**a**+**b**)=div **a**+div **b**,
c) div(u**a**)=u·div **a**+**a**·grad u, where u is a scalar function.

Proof

Here we will prove only property c), while leaving the proof of properties a) and b), as easier ones, to the reader for practice.

Given that

$$u\mathbf{a} = ua_x\mathbf{i} + ua_y\mathbf{j} + ua_z\mathbf{k} = (ua_x)\mathbf{i} + (ua_y)\mathbf{j} + (ua_z)\mathbf{k},$$

ir follows that

$$\begin{aligned}\operatorname{div}(u\mathbf{a}) &= \frac{\partial}{\partial x}(ua_x) + \frac{\partial}{\partial y}(ua_y) + \frac{\partial}{\partial z}(ua_z) = \\ &= u\cdot\frac{\partial a_x}{\partial x} + \frac{\partial u}{\partial x}\cdot a_x + u\cdot\frac{\partial a_y}{\partial y} + \frac{\partial u}{\partial y}\cdot a_y + u\cdot\frac{\partial a_z}{\partial z} + \frac{\partial u}{\partial z}\cdot a_z = \\ &= u\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) + \frac{\partial u}{\partial x}\cdot a_x + \frac{\partial u}{\partial y}\cdot a_y + \frac{\partial u}{\partial z}\cdot a_z = \\ &= u\cdot\operatorname{div}\mathbf{a} + \mathbf{a}\cdot\operatorname{grad}u,\end{aligned}$$

by which the property c) is proven.

Some properties of rotor

a) rot **c**=0,if **c**=$\overrightarrow{\text{const}}$.
b) rot (c**a**)=c rot **a**, c =const.,
c) rot (**a**+**b**)=rot **a**+rot **b**,
d) rot (u**a**)=u rot **a**+ **a**×grad u, where u is a scalar function.

4.2.6 A brief overview of introduced concepts

				vector operations of the I type		vector operations of the II type
u – scalar function	$\to$	grad u – vector			$\to$	$\begin{cases} \operatorname{div}(\operatorname{grad} u) \\ \operatorname{rot}(\operatorname{grad} u) \end{cases}$
$\mathbf{a}$ – vector function	$\to$	div $\mathbf{a}$ – scalar			$\to$	$\operatorname{grad}(\operatorname{div}\mathbf{a})$
		rot $\mathbf{a}$ – vector			$\to$	$\begin{cases} \operatorname{div}(\operatorname{rot}\mathbf{a}) \\ \operatorname{rot}(\operatorname{rot}\mathbf{a}) \end{cases}$

Higher order operations

Let $u = u(x,y,z)$ be a scalar field, then

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} = u'_x\mathbf{i} + u'_y\mathbf{j} + u'_z\mathbf{k}.$$

Let us now determine vector values of II order

a)

$$\begin{aligned} \operatorname{div}\operatorname{grad} u &= \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla u \quad - \quad \text{is a scalar.} \end{aligned} \tag{***}$$

b)

$$\begin{aligned} \operatorname{rot}(\operatorname{grad} u) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = \\ &= \left(\frac{\partial^2 u}{\partial y\partial z} - \frac{\partial^2 u}{\partial z\partial y}\right)\mathbf{i} - \left(\frac{\partial^2 u}{\partial x\partial z} - \frac{\partial^2 u}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2 u}{\partial y\partial x} - \frac{\partial^2 u}{\partial x\partial y}\right)\mathbf{k}. \end{aligned}$$

As, for continuous functions

$$\frac{\partial^2 u}{\partial y\partial z} = \frac{\partial^2 u}{\partial z\partial y},$$
$$\frac{\partial^2 u}{\partial x\partial z} = \frac{\partial^2 u}{\partial z\partial x},$$
$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y},$$

we finally obtain

$$\operatorname{rot}(\operatorname{grad} u) \equiv 0. \tag{4.69}$$

c)

$$\begin{aligned}
\operatorname{grad}(\operatorname{div}\mathbf{v}) &= \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\mathbf{i}+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\mathbf{j}+ \\
&+\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\mathbf{k} = \\
&= \left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}\right)\mathbf{i}+\left(\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial y\partial z}\right)\mathbf{j}+ \\
&+\left(\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}+\frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{k}.
\end{aligned}$$

d)

$$\operatorname{div}(\operatorname{rot}\mathbf{v}) = ?$$

Let

$$\mathbf{v} = v_x\,\mathbf{i}+v_y\,\mathbf{j}+v_z\,\mathbf{k},$$

from where, according to definition (4.46) on page 92, for rot $\mathbf{v}$ we obtain

$$\begin{aligned}
\operatorname{rot}\mathbf{v} = \left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{k} \Rightarrow \\
\Rightarrow \operatorname{rot}\mathbf{v} = a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k} \equiv \mathbf{a},
\end{aligned} \tag{4.70}$$

where we have introduced the following notation

$$\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\equiv a_1,\ \left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\equiv a_2,\ \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\equiv a_3.$$

Further, as

$$\operatorname{div}\mathbf{a} = \frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z},$$

we finally obtain

$$\begin{aligned}
\operatorname{div}(\operatorname{rot}\mathbf{v}) &= \\
&= \left(\frac{\partial^2 v_z}{\partial x\partial y}-\frac{\partial^2 v_y}{\partial x\partial z}\right)+\left(\frac{\partial^2 v_x}{\partial y\partial z}-\frac{\partial^2 v_z}{\partial x\partial y}\right)+\left(\frac{\partial^2 v_y}{\partial x\partial z}-\frac{\partial^2 v_x}{\partial y\partial z}\right) \\
&\equiv 0.
\end{aligned} \tag{4.71}$$

4.2.7 Spatial derivation

The procedure of generalizing the derivative in a direction is called **spatial derivation**, and the result of this procedure is called **spatial derivative**.

Observe a function $\varphi(\mathbf{r})$, which can be a scalar or vector position function.

Let us note in the field of this function a point A and a region $\overline{V}$ (part of the space) bordered by a closed oriented surface S, such that $A \in \overline{V}$. Let

$$\operatorname{mes}\overline{V} = V \tag{4.72}$$

be the measure of the volume of this region, and d$\mathbf{S}$ the vector surface element on the closed oriented surface S.

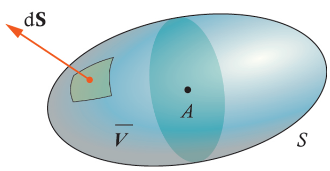


Figure 4.10

Let us further assume that φ is an integrable function on the surface S, that is, that there exist an integral over the closed surface S:

$$I = \iint\limits_S \varphi(\mathbf{r}) \circ \mathrm{d}\mathbf{S}. \tag{4.73}$$

This integral can be a scalar or vector function of the value V, of the region $\overline{V}$, bordered by the closed surface S.

Observe now the value I/V and let the surface "tighten" around the fixed point A, that is, let $V \to 0$. The question now arises as to the existence and determination of the limit value of the quotient I/V.

Definition

We call the **spatial derivative** of the function $\varphi(\mathbf{r})$ the limit value

$$\lim_{V \to 0} \frac{\iint_S \varphi(\mathbf{r}) \circ \mathrm{d}\mathbf{S}}{V} \tag{4.74}$$

if it exists.

If $\varphi(\mathbf{r})$ is a scalar position function, then $\varphi(\mathbf{r}) \circ \mathrm{d}\mathbf{S}$ is a vector, and consequently the spatial derivative is also a vector, which will be denoted by

$$\nabla\varphi(\mathbf{r}) = \lim_{V \to 0} \frac{\iint_S \varphi(\mathbf{r})\, \mathrm{d}\mathbf{S}}{V}. \tag{4.75}$$

It can be proved that this quantity represents the already defined **gradient**

$$\operatorname{grad}\varphi = \nabla\varphi(\mathbf{r}) = \lim_{V \to 0} \frac{\iint_S \varphi(\mathbf{r})\, \mathrm{d}\mathbf{S}}{V}. \tag{4.76}$$

If $\varphi(\mathbf{r})$ is a vector position function

$$\varphi(\mathbf{r}) \equiv \mathbf{v}(\mathbf{r}), \tag{4.77}$$

then, depending on the meaning of the circle-product, we can distinguish two possible cases.

In the first case, where the circle-product represents scalar multiplication, the product $\mathbf{v}\circ\mathrm{d}\mathbf{S}$ represents a scalar, and consequently the spatial derivative is also a scalar, denoted by

$$\nabla\mathbf{v} = \operatorname{div}\mathbf{v} = \lim_{V \to 0} \frac{\iint_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}}{V} \tag{4.78}$$

which will be called **divergence**.

In the second case, where the circle-product represents vector multiplication, the product $\mathbf{v}\times\mathrm{d}\mathbf{S}$ represents a vector, and consequently the spatial derivative is also a vector, denoted by

$$\nabla \times \mathbf{v} = \operatorname{rot}\mathbf{v} = \lim_{V \to 0} \frac{\iint_S \mathbf{v} \times \mathrm{d}\mathbf{S}}{V} \tag{4.79}$$

and defining a value called **rotor**.

From previous definitions of *gradient*, *divergence* and *rotor*, their independence from the choice of coordinate system follows, which has already been mentioned when they were defined in the previous chapter.

Divergence and rotor of the vector function of a constant direction

It is of special interest to find the expression for the divergence of the vector function of a constant direction.

Observe one such vector $\mathbf{c}$

$$\mathbf{c} = c\mathbf{c}_0, \quad \text{where } \mathbf{c}_0 \text{ is the unit vector of a constant direction.} \tag{4.80}$$

Further, according to the definition

$$\operatorname{div}\mathbf{c} = \lim_{V\to 0}\frac{\iint_S \mathbf{c}\,\mathrm{d}\mathbf{S}}{V} = \lim_{V\to 0}\frac{\iint_S c\,\mathrm{d}\mathbf{S}}{V}\cdot\mathbf{c}_0 = \operatorname{grad} c\cdot\mathbf{c}_0. \tag{4.81}$$

Using this relation, we can write an analytical expression for the divergence in a rectangular coordinate system.

Observe a vector function, expressed in one of the possible ways[10]

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i. \tag{4.82}$$

According to the relation above

$$\begin{aligned}\operatorname{div}\mathbf{v} &= \operatorname{div}(v_x\mathbf{i}) + \operatorname{div}(v_y\mathbf{j}) + \operatorname{div}(v_z\mathbf{k}) = \\ &= \mathbf{i}\cdot\operatorname{grad} v_x + \mathbf{j}\cdot\operatorname{grad} v_y + \mathbf{k}\cdot\operatorname{grad} v_z = \\ &= \sum_{i=1}^{3}\mathbf{e}_i\cdot\operatorname{grad} v_i \equiv \mathbf{e}_i\cdot\operatorname{grad} v_i.\end{aligned} \tag{4.83}$$

As

$$\operatorname{grad} v_i = \frac{\partial v_i}{\partial x}\mathbf{i} + \frac{\partial v_i}{\partial y}\mathbf{j} + \frac{\partial v_i}{\partial z}\mathbf{k}, \tag{4.84}$$

we finally obtain

$$\operatorname{div}\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{4.85}$$

Thus, the same expression as in the previous chapter for Cartesian coordinates.

In the case of rotor, it is useful to determine it for the vector of a constant direction

$$\mathbf{c} = c\,\mathbf{c}_0, \quad c = |\mathbf{c}| \quad \mathbf{c}_0 = \overrightarrow{\text{const.}} \tag{4.86}$$

From the definition we obtain

$$\operatorname{rot}\mathbf{c} = \lim_{V\to 0}\frac{\iint_S \mathbf{c}\times\mathrm{d}\mathbf{S}}{V} = \lim_{V\to 0}\frac{\iint_S c\mathrm{d}\mathbf{S}}{V}\times\mathbf{c}_0 = \operatorname{grad} c\times\mathbf{c}_0. \tag{4.87}$$

If we apply this to some vector function

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \tag{4.88}$$

we obtain

$$\begin{aligned}\operatorname{rot}\mathbf{a} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_x & a_y & a_z\end{vmatrix} = \\ &= (\operatorname{grad} a_x\times\mathbf{i}) + (\operatorname{grad} a_y\times\mathbf{j}) + (\operatorname{grad} a_z\times\mathbf{k}) = \\ &= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k},\end{aligned} \tag{4.89}$$

thus, the same expression as in the previous chapter for Cartesian coordinates.

[10] Note that when writing this expression, we used the addition convention, according to which $\sum_{i=1}^{3} v_i\mathbf{e}_i \equiv v_i\mathbf{e}_i$, i.e. addition by repeated indices is performed.

4.2.8 Integral theorems

In this part we will outline several theorems (Stokes'[11], Green's[12], Gauss's[13]) which are very often used in integral calculus and its applications.[14]

Stokes's theorem

If the projections $v_x(x,y,z)$, $v_y(x,y,z)$ and $v_z(x,y,z)$, of a vector function $\mathbf{v}(\mathbf{r})$ are continuous, as well as their corresponding partial derivatives, on the surface S, which is closed by the spatial curve C, then

$$\oint_C \mathbf{v}\,\mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}S \left(= \iint_S (\nabla\times\mathbf{v})\cdot\mathrm{d}\mathbf{S} = \iint_S \mathrm{rot}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S} \right), \tag{4.90}$$

where $\mathbf{n}$ is the unit vector of the normal to the observed surface.

Green's theorem

If for a scalar function Φ there exists a line integral along a closed line C, and if gradΦ is a continuous function in the region S bordered by the curve C, then

$$\oint_C \Phi\,\mathrm{d}\mathbf{r} = \iint_S (\mathbf{n}\times\nabla\Phi)\,\mathrm{d}S \left(= \iint_S \mathrm{d}\mathbf{S}\times\nabla\Phi \right). \tag{4.91}$$

Gauss's theorem

If for a vector function $\mathbf{v}(\mathbf{r})$ there exists a surface integral along a closed surface S, which represents the border of a region V and if div$\,\mathbf{v}$ is a continuous function in this region, then

$$\iiint_V \nabla\cdot\mathbf{v}\,\mathrm{d}V = \iint_S \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S \left(= \iint_S \mathbf{v}\cdot\mathrm{d}\mathbf{S} \right). \tag{4.92}$$

This theorem is also known as the divergence theorem or the Gauss–Ostrogradsky theorem.[15].

The mean value theorem

1. If $f(x,y)$ is a continuous function on a closed and limited region σ in the x,y plane, then there exists at least one point $(x_o,y_o)\in\sigma$ such that $\iint_\sigma f(x,y)\,\mathrm{d}\sigma = f(x_o,y_o)\cdot P$, where P is the area of region σ.
2. If $f(x,y,z)$ is a continuous function on a closed and limited region σ in space, then there exists at least one point $(x_o,y_o,z_o)\in\sigma$ such that

$$\iiint_\sigma f(x,y,z)\,\mathrm{d}\sigma = f(x_o,y_o,z_o)\cdot V, \tag{4.93}$$

where V is the volume of region σ.

[11] Stokes, George Gabriel (1819–1903), Irish mathematician and physicist. He is known for his contributions to the theory of infinite series as well as contributions to fluid mechanics (Navier-Stokes equations), geodesy and optics.

[12] Green, George (1793-1841), English mathematician. His work pertains to the theory of potentials related to electricity and magnetism, as well as to oscillations, waves and the theory of elasticity.

[13] Gauss, Carl Friedrich (1777–1855), a great German mathematician. His work is of fundamental importance for algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, astronomy, geodesy, electromagnetism and theoretical mechanics.

[14] Proof of these theorems, due to limited space, is not given, but they are outlined here due to their importance.

[15] Остроградский, Михаил Васил мекиевич (1801-1862). Famous Russian mathematician and mechanics scientist.

4.3 Examples of some fields of interest for physics and engineering

We will now present some examples of potential fields that are of particular interest for physics and engineering.

Attraction of two points in the field of gravitational force

Newton's gravitational force is defined by the expression

$$\mathbf{F} = -\gamma \frac{m m_0}{r^2} \mathbf{r}_0, \tag{4.94}$$

where m and m_0 are masses being attracted, and γ is the universal gravitational constant.
In the previous expression we have introduced the following notation (Fig. 4.11)

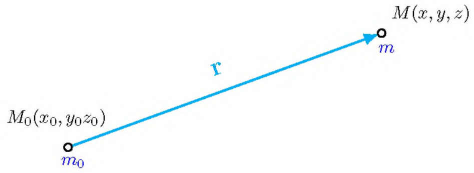


Figure 4.11: Attraction of two points.

$$\mathbf{r}_0 = \frac{\mathbf{r}}{r}, \quad \text{where} \quad \mathbf{r} = \overrightarrow{M_0 M}. \tag{4.95}$$

The question is, whether there exists a potential for a force defined in such a manner? As masses m and m_0, as well as γ, are constant, we can substitute them by a single constant, for example c, and we can then represent the force in the following form

$$\mathbf{F} = -\frac{c}{r^2} \mathbf{r}_0. \tag{4.96}$$

We have shown earlier (see p. 93) that if there exists a scalar function U, such that $\mathbf{F} = \text{grad} U$, then the vector field $\mathbf{F}$ is potential. Thus, such a scalar function $U = U(r)$ should be determined. Given that

$$\text{grad}\, U = \frac{\mathrm{d}U}{\mathrm{d}r} \mathbf{r}_0 \quad \text{i} \quad \mathbf{F} = -\frac{c}{r^2} \mathbf{r}_0$$

from $\mathbf{F} = \text{grad}\, U$ we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{c}{r^2} \quad \text{that is} \quad \mathrm{d}U = -\frac{c}{r^2} \mathrm{d}r$$

and then

$$U = \frac{2c}{r} + c_1.$$

Based on these results, it can be concluded that the gravitational force, defined by (4.94), can be represented by

$$\mathbf{F} = \text{grad}\, U, \tag{4.97}$$

i.e. the force is potential, and its potential can be determined by the expression

$$U = \frac{C}{r}. \tag{4.98}$$

This potential is also known in literature as **Newton's potential**. Here, we assumed that $2c = C$ and $c_1 = const. = 0$, which does not affect the generality of the previously derived expression.

It has been shown that $\Delta(1/r) = 0$.

Thus, the potential U satisfies the Laplace equation

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{4.99}$$

Functions that satisfy the Laplace equation are called **harmonic functions**. Thus, Newton's potential is a harmonic function.

Plane task. Logarithmic potential

Consider now the force in a plane by which some point O attracts a material point M.

Suppose this force of attraction is given by the expression

$$\mathbf{F} = -\frac{2c}{r}\frac{\mathbf{r}}{r} \tag{4.100}$$

where $r = |\mathbf{r}|$ is the magnitude of the position vector $\mathbf{r}$.

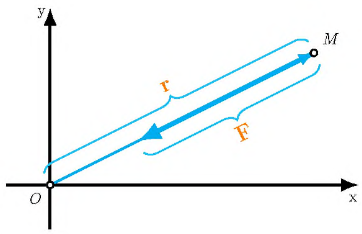


Figure 4.12: Force of attraction - logarithmic potential.

Once again, the question is whether there exists a potential U for the force defined in such a manner. Similar to the previous case, we start from:

$$\begin{aligned} X &= \frac{\partial U}{\partial x} = -\frac{2c}{r^2}x, \\ Y &= \frac{\partial U}{\partial y} = -\frac{2c}{r^2}y. \end{aligned} \tag{4.101}$$

From here we obtain

$$U = -2c\int \frac{x}{r^2}\,dx + f(y).$$

Further, given that

$$r^2 = x^2 + y^2, \quad \text{it follows that} \quad 2x\,\mathrm{d}x = 2r\,\mathrm{d}r,$$

and we obtain the potential U

$$U = -2c\ln r + f(y).$$

Let us now find the partial derivative by y:

$$\begin{aligned} \frac{\partial U}{\partial y} &= \frac{\partial}{\partial y}(-2c\ln r) + \frac{\mathrm{d}f}{\mathrm{d}y} = \\ &= -2c\frac{\partial}{\partial r}(\ln r)\frac{\partial r}{\partial y} + \frac{\mathrm{d}f}{\mathrm{d}y} = -2c\frac{y}{r^2} + \frac{\mathrm{d}f}{\mathrm{d}y}. \end{aligned} \tag{4.102}$$

Thus, according to (4.101) (condition for Y), we can conclude that $\frac{df}{dy} = const$. We shall assume here also that this constant is equal to 0, and we thus obtain the potential

$$U = -2c\ln r. \tag{4.103}$$

R Note that in the case when several points attract one point, the potential is

$$U = U(x,y,z) = \sum_{i=1}^{n} \frac{m_i}{r_i} =$$
$$= \sum_{i=1}^{n} \frac{m_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}}.$$

Newton's force by which a homogeneous spherical shell attracts a material point

Consider now a homogeneous spherical shell, of radius R, which attracts a material point (Fig. 4.13). Let us first determine the force by which an infinitesimal part of the sphere ($\mathrm{d}S$) attracts the observed point

$$\mathrm{d}\mathbf{F} = \frac{k^2 m_1 \mathrm{d}m_2}{r^2} \frac{\mathbf{r}}{r}. \tag{4.104}$$

Given that $\mathrm{d}m_2 = \rho \mathrm{d}V = \rho \cdot d \cdot \mathrm{d}S$, for unit thickness $d = 1$ we obtain $\mathrm{d}m_2 = \rho \mathrm{d}S$. There is always one straight line through the observed point M and the center of the sphere O. In our case, let it be the z – axis, i.e. $\overline{OM} = z$.

Let us denote by r the distance between the point M and the center of the surface $\mathrm{d}S$ (the point where the force of attraction acts). This distance is, according to the figure (application of the cosine theorem)

$$r^2 = R^2 + z^2 - 2zR\cos\theta. \tag{4.105}$$

For the magnitude of the force we now obtain

$$\mathrm{d}F = \frac{k^2 m_1 \rho \mathrm{d}S}{R^2 + z^2 - 2zR\cos\theta}. \tag{4.106}$$

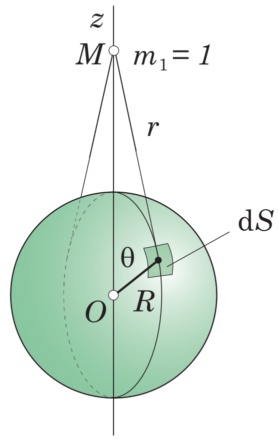


Figure 4.13: Spherical shell - material point M.

Projecting this force on the z – axis, and taking into account the symmetry, we can notice that the projections on the normal direction, perpendicular to z, are mutually canceled, so only the Z – projection should be taken into account (see Fig. 4.14).

(a) Decomposition of force. (b) Distance of shell element from point.

Figure 4.14: Projection of force on the z - axis.

$$\mathrm{d}Z = -\mathrm{d}F \cdot \cos\beta. \tag{4.107}$$

Further, from the relation $z = r\cos\beta + R\cos\theta$ (see Fig. 4.14), we obtain

$$\cos\beta = \frac{z - R cos\theta}{r}, \tag{4.108}$$

and thus finally the projection of the elementary force $\mathrm{d}F$ on the $z-$axis

$$\begin{aligned}\mathrm{d}Z &= -\frac{k^2 m_1 \rho \mathrm{d}S}{R^2 + z^2 - 2zR\cos\theta} \cdot \frac{z - R\cos\theta}{r} = \\ &= -\frac{k^2 m_1 \rho dS (z - R\cos\theta)}{(R^2 + z^2 - 2zR\cos\theta)^{3/2}}.\end{aligned} \tag{4.109}$$

The total force can then be determined by integrating the above expression over the whole sphere S

$$Z = -\int_S \frac{k^2 m_1 \rho (z - R\cdot\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \mathrm{d}S. \tag{4.110}$$

Let us now express the elementary surface $\mathrm{d}S$ in spherical coordinates

$$\mathrm{d}S = R^2 \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta, \tag{4.111}$$

and consequently obtain the expression

$$Z = k^2 m_1 \rho R^2 \int_0^{\pi} \int_0^{2\pi} \frac{(R\cos\theta - z)\cdot \sin\theta \, \mathrm{d}\varphi \mathrm{d}\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}. \tag{4.112}$$

Further, given that $r^2 = R^2 + z^2 - 2Rz\cos\theta$ (see eq. (4.105)), where R, as the radius, and z, as the distance between the fixed point M and the center of the sphere, are fixed distances, we obtain by differentiating that $r\,\mathrm{d}r = Rz\sin\theta\,\mathrm{d}\theta$, that is

$$R\cos\theta - z = -\frac{r^2 + z^2 - R^2}{2z}, \tag{4.113}$$

so that

$$\begin{aligned}\int_0^{\pi} \frac{(R\cos\theta - z)\sin\theta\,\mathrm{d}\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} &= -\frac{1}{2Rz^2}\int_{|z-R|}^{z+R} \left(1 + \frac{z^2 - R^2}{r^2}\right)\mathrm{d}r = \\ &= -\frac{1}{2Rz^2}\left[r - \frac{z^2 - R^2}{r}\right]_{|z-R|}^{z+R}.\end{aligned} \tag{4.114}$$

Let us now analyze this result. As $\theta = 0 \quad \Rightarrow r = |z - R|$, two cases should be distinguished. **First**, when **the point is outside the sphere** then $z > R$, so that $|z - R| = z - R$, from where it follows that

$$Z = -\frac{k^2 m_1 4\pi R^2 \rho}{z^2} = -\frac{k^2 m_1 m_2}{z^2}. \tag{4.115}$$

Here we assumed that the volume of a sphere of unit thickness $V = 4\pi R^2 \cdot d = 4\pi R^2$, where d is the thickness, which is equal to 1, so that the mass is $m_2 = \rho V = 4\pi\rho R^2$. And the **second** case, when **the point is inside** the sphere. In that case $z < R$, so that $|z - R| = R - z$, from where it follows that $Z = 0$.

Attraction of a point by a line body

Let us determine the potential of a homogeneous line L, which coincides with the z – axis. As the observed point M and the given line L determine a plane, let us assume that plane is the xOz plane, see Fig. 4.15.

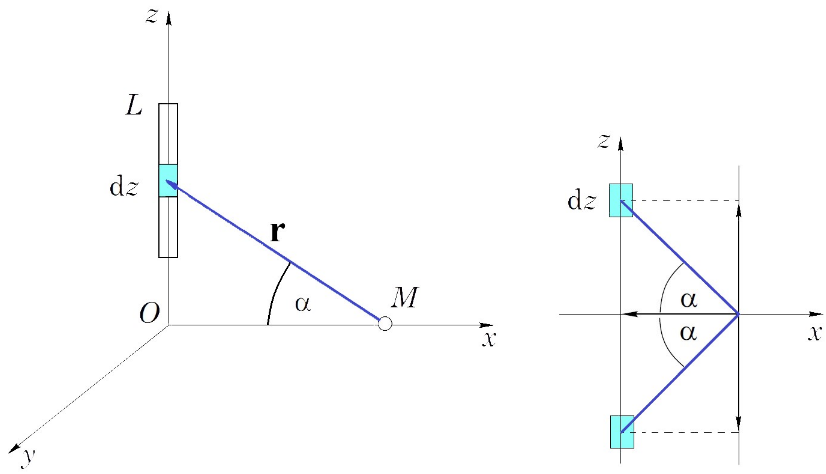


Figure 4.15: Attraction of a point by a line body.

Let us first determine the force by which the line attracts the point. The elementary force is

$$|\mathrm{d}\mathbf{F}| = k\frac{\mathrm{d}z}{r^2} = k\frac{\mathrm{d}z}{x^2+z^2}. \tag{4.116}$$

R Note that we assumed that the force is proportional to the masses, and inversely proportional to the square of the distance. However, given that $\mathrm{d}m = \rho \mathrm{d}V = \rho P \mathrm{d}z$ and $P = 1$, we obtained that the force depends on $\mathrm{d}z$, while we denoted all other constants with one letter – k.

Further, similarly as in the previous example, due to the symmetry of the projections (see Fig. 4.15), once again only the X – projection needs to be determined

$$\mathrm{d}X = \mathrm{d}F\,\cos\alpha = -k\frac{\mathrm{d}z}{r^2}\cdot\frac{x}{r}. \tag{4.117}$$

The total projection is

$$X = -\int_{-\infty}^{+\infty} k\frac{x}{r}\cdot\frac{\mathrm{d}z}{r^2}. \tag{4.118}$$

Further, given that

$$\mathrm{tg}\alpha = \frac{z}{x} \quad \Rightarrow z = x\,\mathrm{tg}\alpha \quad \Rightarrow \mathrm{d}z = \frac{x}{\cos^2\alpha}d\alpha, \quad x = const, \tag{4.119}$$

and

$$r^3 = (x^2+z^2)^{3/2} = x^3(1+\mathrm{tg}^2\alpha)^{3/2} = \frac{x^3}{\cos^3\alpha}, \tag{4.120}$$

for X we obtain

$$X = -k\int_{+\frac{\pi}{2}}^{-\frac{\pi}{2}} x\cdot\frac{\cos^3\alpha}{x^3}\,\frac{x}{\cos^2\alpha}\,\mathrm{d}\alpha. \tag{4.121}$$

Give that $x = const$, we further obtain

$$X = -k\frac{1}{x}\int_{+\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos\alpha\, d\alpha = -\frac{2k}{x}. \tag{4.122}$$

We can now look for the potential. If the potential U exists, it must satisfy the relation

$$X = \frac{\partial U}{\partial x} = -\frac{2k}{x}, \tag{4.123}$$

from where we finally obtain

$$U = -2k\ln x \quad \text{ili} \quad U = 2k\ln\frac{1}{x}. \tag{4.124}$$

Theorem 10

An arbitrary vector field **F**, unambiguous, continuous, and bounded, can be decomposed into the sum of a *potential* and an *irrotational* vector field in the form

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{A},$$

where $\nabla \cdot \mathbf{A} = 0$.

The scalar function U is called the **scalar potential**, and the vector function **A** is called the **vector potential** of the vector field **F**.

This theorem is known in literature as Helmholtz's[16] theorem. We have stated the theorem because of its importance. However, as the proof is relatively complex, we will not present it, but rather refer the reader to references: [**4**] (p. 79), [**29**] (p. 50).

4.4 Generalized coordinates

The position of a point in three-dimensional Euclidean space is determined in relation to a predetermined point O, which is called the pole or origin, by a *position vector* **r**. In a Cartesian rectangular coordinate system $Oxyz$, with origin in pole O, the position of a point is determined by the *point coordinates* (x, y, z). Orthogonal projections of the end of the position vector on the axes of this coordinate system coincide with the coordinates of the point, and thus the coordinates of the position vector **r** coincide with these coordinates x, y, z:

$$\mathbf{r} = \{x, y, z\} \quad \text{or} \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{4.125}$$

However, the position of a point in space can also be determined using some three mutually independent parameters q^1, q^2, q^3 (1, 2 and 3 here are not powers but rather parameter designations !!!), or shorter q^i (i=1,2,3). When the parameter q^i receives all possible values, and one and only one ordered set of three numbers (q^1, q^2, q^3) corresponds to each point of space, and conversely, one and only one point in space (three-dimensional) corresponds to each set of thw three numbers (q^1, q^2, q^3), then the *parameters* q^i are called **general** or **generalized** coordinates of the point. The position vector can now be represented by the generalized coordinates:

$$\mathbf{r} = \mathbf{r}(q^1, q^2, q^3), \quad \text{or shortly} \quad \mathbf{r} = \mathbf{r}(q^i). \tag{4.126}$$

[16]Hermann von Helmholtz (1821–1894), German physicist. He is known for very important works in the field of thermodynamics, hydrodynamics and acoustics.

Cartesian coordinates of the vector can now be expressed in the following form:

$$\begin{aligned} x &= x(q^i), \\ y &= y(q^i), \\ z &= z(q^i). \end{aligned} \tag{4.127}$$

It follows from the requirement for one-to-one correspondence between points in space and the coordinates q^i that to each point with coordinates (x,y,z) three numbers q^i *must correspond*, such that:

$$q^i = q^i(x,y,z), \qquad \frac{\partial(q^1,q^2,q^3)}{\partial(x,y,z)} \neq 0. \tag{4.128}$$

Thus, equations (4.127) always satisfy the conditions necessary for solving them for q^i.

The equations (4.127) and (4.128) are equations of coordinate transformation. These transformations are mutually reciprocal – inverse. Let us now define coordinate lines and coordinate surfaces.

Definition

Coordinate lines – represent the geometric location of points obtained when two coordinates are constant and the third changes.

Coordinate lines can be straight or curved, and depending on that, we distinguish rectilinear and curvilinear coordinate systems (Fig. 4.16).

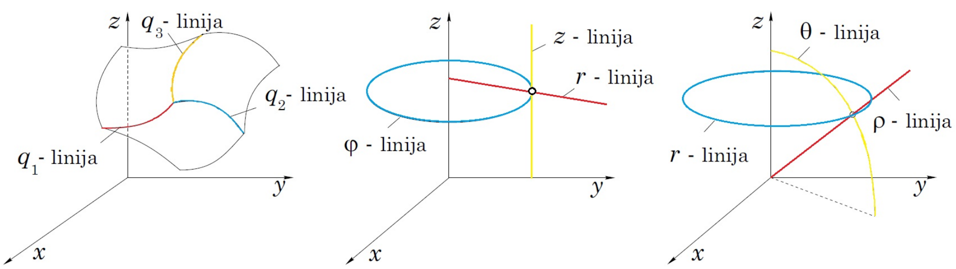


Figure 4.16: Coordinate lines.

Definition

Coordinate surfaces – represent the geometric location of points obtained when two coordinates change and the third remains constant.

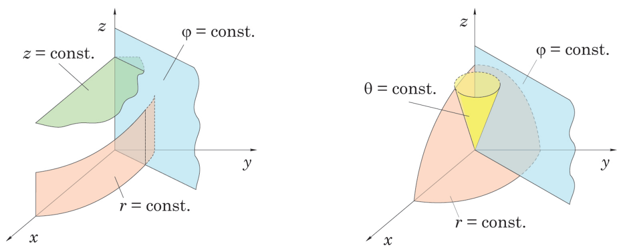


Figure 4.17: Coordinate surfaces

To each point of space three coordinate surfaces correspond, i.e. each point represents an intersection of three coordinate surfaces. A coordinate line is located at the intersection of two coordinate surfaces (Fig. 4.17). As three such, in general, curved lines, pass through each point, they form a curvilinear coordinate system.

For each point in space tangents to coordinate lines at **that point** can be drawn, so that they are oriented in the direction in which the values of q^i grow.

The base vectors of this coordinate system are the tangent vectors.

R Note that in the case of a curvilinear coordinate system, the base vectors change from one point to another, unlike in rectilinear coordinate systems.

Coordinate systems, in relation to which the position of points in space is determined, are also called reference systems. Depending on the type of the coordinate line, coordinates can be

- rectilinear (Fig. 4.18) and
- curvilinear (Fig. 4.16).

If the coordinate surfaces, i.e. the coordinate lines, are mutually perpendicular at all points in space, they form an **orthogonal** (rectangular) coordinate system. In the special case of the Cartesian coordinate system, the position of a point in space is determined by the position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{4.129}$$

It is obvious that

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial x}, \quad \mathbf{j} = \frac{\partial \mathbf{r}}{\partial y}, \quad \mathbf{k} = \frac{\partial \mathbf{r}}{\partial z}. \tag{4.130}$$

Thus, unit vectors (orts) **i**, **j** and **k** can be obtained as partial derivatives of the position vector by point coordinates.

In the case of generalized coordinates, the position of the same point is determined by the same position vector **r**, but expressed in relation to the curvilinear coordinates q^i (see 4.126). Then the base vectors of that coordinate system are determined (as in the case of Cartesian coordinates (4.130)) by

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial q^i}, \quad (i = 1,2,3). \tag{4.131}$$

Thus, the vectors $\mathbf{g}_i$ are the **base vectors** of the system of generalized coordinates q^i and they have the directions of the tangents to the coordinate lines. That they are base vectors is concluded on the basis of their linear independence.

We notice that the indices that denote the base vectors $\mathbf{g}_i$ and $\mathbf{e}_i$ are at the bottom position.

If coordinates of the basic vectors are expressed in relation to Cartesian rectangular coordinates, the magnitudes of these vectors are determined by the following expressions

$$|\mathbf{g}_i| = \sqrt{\left(\frac{\partial x}{\partial q^i}\right)^2 + \left(\frac{\partial y}{\partial q^i}\right)^2 + \left(\frac{\partial z}{\partial q^i}\right)^2} \equiv h_i, \quad (i = 1,2,3), \tag{4.132}$$

where h_i are the so called **Lamé** [17] or metric **coefficients**.

If we denote the unit vectors in the directions of these vectors by $\mathbf{e}_i$, we obtain

$$\mathbf{e}_i = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} \quad \Rightarrow \quad \mathbf{g}_i = h_i \mathbf{e}_i. \tag{4.133}$$

[17] Gabriel Lamé (1795-1870), French mathematician and engineer. He made important contributions to analytical geometry and analytical mechanics, as well as to the theory of elasticity and the theory of heat conduction.

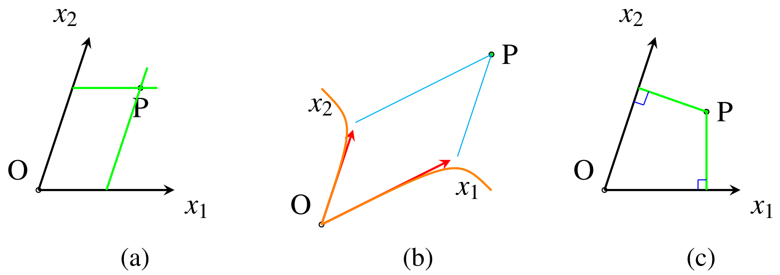


Figure 4.18: Coordinate lines forming an acute angle.

In addition to this set of unit vectors, we can also observe a set of vectors gradq^i, which are in the directions of the normals to the coordinate surfaces, and are defined by

$$\mathbf{E}^i = \frac{\nabla q^i}{|\nabla q^i|}. \tag{4.134}$$

Vectors $\mathbf{E}^i$ are therefore unit vectors of normals to the coordinate surfaces. Thus, at each point in space there are two sets of unit vectors: $\mathbf{e}_i$ – unit vectors in the directions of the tangents, and $\mathbf{E}^i$ – unit vectors of the normals to the coordinate surfaces (Fig. 4.19). If the coordinate system is orthogonal these two sets coincide.

Let us observe an arbitrary vector $\mathbf{v}$ and represent it by these two systems

$$\begin{aligned} \mathbf{v} = & \\ & = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3 = \sum_{i=1}^{3} v^i\mathbf{e}_i = \\ & = V_1\mathbf{E}^1 + V_2\mathbf{E}^2 + V_3\mathbf{E}^3 = \sum_{i=1}^{3} V_i\mathbf{E}^i. \end{aligned} \tag{4.135}$$

This vector can also be represented by the base vectors $\mathbf{g}_i = \dfrac{\partial \mathbf{r}}{\mathbf{q^i}}$:

$$\begin{aligned} \mathbf{v} = & \\ & = \sum_{i=1}^{3} c^i\mathbf{g}_i = \sum_{i=1}^{3} c^i \frac{\partial \mathbf{r}}{\partial q^i} = c^1\frac{\partial \mathbf{r}}{\partial q^1} + c^2\frac{\partial \mathbf{r}}{\partial q^2} + c^3\frac{\partial \mathbf{r}}{\partial q^3} \\ & = \sum_{i=1}^{3} C_i\nabla q^i = C_1\nabla q^1 + C_2\nabla q^2 + C_3\nabla q^3. \end{aligned} \tag{4.136}$$

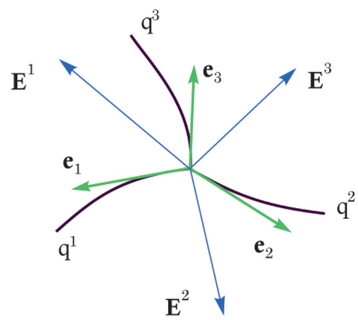


Figure 4.19: Contravariant and covariant base.

The coordinates obtained by decomposing this vector are called
c^i – **contravariant** and
C_i – **covariant**, respectively.

4.4.1 Arc and volume elements

The length of an arc element (Fig. 4.20) is determined by the relation

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r}. \tag{4.137}$$

The differential of this vector can be expressed in one of the following ways (depending on whether the coordinates are Cartesian or some other - generalized coordinates)

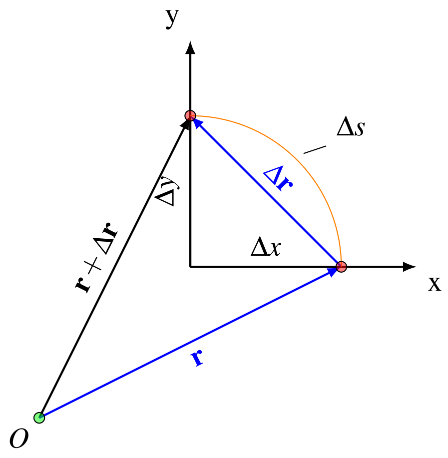


Figure 4.20: Arc element.

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} + \mathrm{d}z\mathbf{k} = \\ &= \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial q^i}\mathrm{d}q^i = \sum_{i=1}^{3} \mathbf{g}_i \mathrm{d}q^i = \\ &= \sum_{i=1}^{3} h_i \mathbf{e}_i \mathrm{d}q^i . \end{aligned} \tag{4.138}$$

If the Cartesian coordinates are denoted by x^1, x^2 and x^3, instead of x, y and z, respectively, for the arc element (via Cartesian coordinates) we obtain

$$\mathrm{d}s^2 = \sum_{i,j=1}^{3} \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = \sum_{i=1}^{3} \mathrm{d}x^i \mathrm{d}x^i \tag{4.139}$$

where δ_{ij} is the Kronecker delta symbol.

We can then determine $\mathrm{d}x^i$, according to (4.127), as

$$x^i = x^i(q^j) \quad \Rightarrow \quad \mathrm{d}x^i = \sum_{j=1}^{3} \frac{\partial x^i}{\partial q^j}\mathrm{d}q^j , \tag{4.140}$$

and obtain the arc element

$$\begin{aligned} ds^2 &= \sum_{i,j}^{3} \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} \delta_{ij} \frac{\partial x^i}{\partial q^k} \cdot \frac{\partial x^j}{\partial q^l} \mathrm{d}q^k \mathrm{d}q^l = \\ &= \sum_{k,l=1}^{3} \left(\sum_{i,j=1}^{3} \delta_{ij} \frac{\partial x^i}{\partial q^k} \cdot \frac{\partial x^j}{\partial q^l} \right) \mathrm{d}q^k \mathrm{d}q^l = \\ &= \sum_{k,l=1}^{3} g_{kl}\mathrm{d}q^k \mathrm{d}q^l . \end{aligned} \tag{4.141}$$

On the other hand, it follows from (4.138) that

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \sum_{k,l=1}^{3} \mathbf{g}_k \cdot \mathbf{g}_l \mathrm{d}q^k \mathrm{d}q^l . \tag{4.142}$$

Comparing (4.142) and (4.141) we can conclude that $\mathbf{g}_k \cdot \mathbf{g}_l = g_{kl}$.

Definition

The variables g_{kl} defined by the relations

$$g_{kl} = \mathbf{g}_k \cdot \mathbf{g}_l = \sum_{i,j=1}^{3} \delta_{ij} \frac{\partial x^i}{\partial q^k} \cdot \frac{\partial x^j}{\partial q^l} \tag{4.143}$$

represent the **base metric tensor** or base space tensor.

Note that for an orthogonal coordinate system $g_{ij} = h_i^2 \delta_{ij}$. Namely, in that case we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{4.144}$$

or, in the expanded format

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \\ \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \end{aligned} \tag{4.145}$$

and thus in an orthogonal coordinate system, for the arc element we obtain

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \\ &= \sum_{i,j=1}^{3} h_i dq^i h_j dq^j \mathbf{e}_i \cdot \mathbf{e}_j = \sum_{i,j=1}^{3} \delta_{ij} h_i dq^i h_j dq^j = \\ &= \sum_{i=1}^{3} \left(h_i dq^i\right)^2. \end{aligned} \tag{4.146}$$

Finally, let us express the volume element (Fig. 4.21) by generalized orthogonal coordinates

$$\begin{aligned} \mathrm{d}V &= \left| \left(h_1 dq^1 \mathbf{e}_1\right) \cdot \left(h_2 dq^2 \mathbf{e}_2\right) \times \left(h_3 dq^3 \mathbf{e}_3\right) \right| = \\ &= h_1 h_2 h_3 dq^1 dq^2 dq^3 \left(\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3\right) = h_1 h_2 h_3 dq^1 dq^2 dq^3. \end{aligned} \tag{4.147}$$

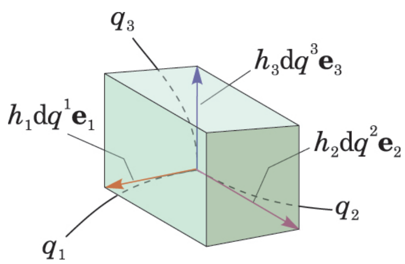


Figure 4.21: Volume element.

4.4.2 Gradient, divergence, rotor and Laplacian - expressed by generalized coordinates

The gradient of an arbitrary function U can be expressed in generalized coordinates as

$$\begin{aligned} \mathrm{grad}U &= \nabla U = \\ &= \frac{1}{h_1} \cdot \frac{\partial U}{\partial q^1} \mathbf{e}_1 + \frac{1}{h_2} \cdot \frac{\partial U}{\partial q^2} \mathbf{e}_2 + \frac{1}{h_3} \cdot \frac{\partial U}{\partial q^3} \mathbf{e}_3 = \\ &= \sum_{i=1}^{3} \frac{1}{h_i} \cdot \frac{\partial U}{\partial q^i} \mathbf{e}_i. \end{aligned} \tag{4.148}$$

Divergence is given by the expression

$$\begin{aligned} \mathrm{div}\mathbf{A} &= \nabla \cdot \mathbf{A} = \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} \left(h_2 h_3 A_1\right) + \frac{\partial}{\partial q^2} \left(h_3 h_1 A_2\right) + \frac{\partial}{\partial q^3} \left(h_1 h_2 A_3\right) \right]. \end{aligned} \tag{4.149}$$

Rotor is given by the expression

$$\mathrm{rot}\mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \cdot \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \dfrac{\partial}{\partial q^1} & \dfrac{\partial}{\partial q^2} & \dfrac{\partial}{\partial q^3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}. \tag{4.150}$$

Finally, let us represent the Laplacian ($\Delta = \nabla^2$) with respect to generalized coordinates

$$\Delta U = \tag{4.151}$$
$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2 h_3}{h_1}\cdot\frac{\partial U}{\partial q^1}\right)+\frac{\partial}{\partial q^2}\left(\frac{h_3 h_1}{h_2}\cdot\frac{\partial U}{\partial q^2}\right)+\frac{\partial}{\partial q^3}\left(\frac{h_1 h_2}{h_3}\cdot\frac{\partial U}{\partial q^3}\right)\right].$$

4.5 Special coordinate systems

1. CYLINDRICAL (ρ, φ, z)

The relation between Cartesian and cylindrical coordinates is given by (Fig. 4.22):

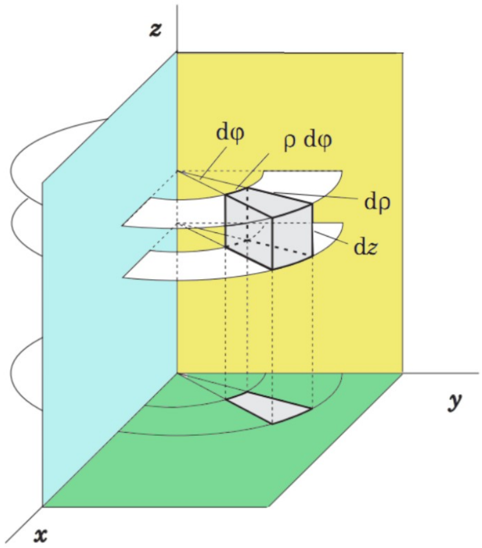


Figure 4.22: Cylindrical coordinate system.

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z,$$
$$\text{where} \quad \rho > 0, \quad 0 \le \varphi < 2\pi, \quad -\infty < z < +\infty.$$

Let us now show on the example of these coordinates how the Lamé coefficients are determined. Let us first express the differentials $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ by the new coordinates

$$dx = \cos\varphi\, d\rho + (-\sin\varphi)\,\rho\, d\varphi,$$
$$dy = \sin\varphi\, d\rho + \cos\varphi\,\rho\, d\varphi,$$
$$dz = dz.$$

Let us then express the arc element

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 =$$
$$= \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2, \tag{4.152}$$

and then, comparing with (4.141), we can conclude that

$$h_\rho = 1 \quad h_\varphi = \rho \quad h_z = 1.$$

2. SPHERICAL (r, θ, φ)

In this case, the relation between coordinates, according to Fig. 4.23, is

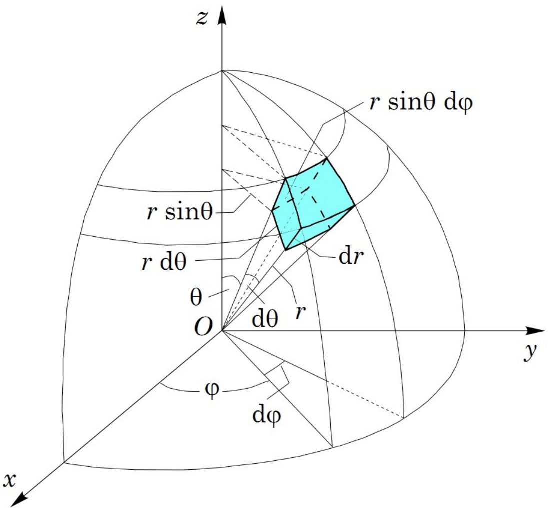


Figure 4.23: Spherical coordinate system.

$$x = r\sin\theta\,\cos\varphi, \quad y = r\sin\theta\,\sin\varphi, \quad z = r\cos\theta,$$
$$\text{where} \quad r > 0, \quad 0 \le \varphi < 2\pi, \quad 0 < \theta < \pi.$$

In a similar way as in the previous case we obtain

$$h_r = 1 \quad h_\varphi = r\sin\theta \quad h_\theta = r.$$

3. PARABOLICAL-CYLINDRICAL (u, v, z)

$$x = \frac{1}{2}\left(u^2 - v^2\right), \quad y = uv, \quad z = z$$
$$\text{where} \quad -\infty < u < \infty, \quad v \ge 0, \quad -\infty < z < \infty$$
$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_z = 1,$$

while the relation with cylindrical coordinates is given by

$$u = \sqrt{2\rho}\cos\frac{\varphi}{2}, v = \sqrt{2\rho}\sin\frac{\varphi}{2}, z = z.$$

4. PARABOLOID (u, v, φ)

$$x = uv\cos\varphi, \quad y = uv\sin\varphi, \quad z = \frac{1}{2}\left(u^2 - v^2\right)$$
$$\text{where} \quad u \geq 0, \quad v \geq 0, \quad 0 \leq \varphi < 2\pi$$
$$\text{i} \quad h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\varphi = uv.$$

5. ELLIPTICAL-CYLINDRICAL (u, v, z)

$$x = a\,\mathrm{ch}\,u\cos v, \quad y = a\,\mathrm{sh}\,u\sin v, \quad z = z$$
$$\text{where} \quad u \geq 0, \quad 0 \leq v < 2\pi, \quad -\infty < z < \infty$$
$$\text{i} \quad h_u = h_v = a\sqrt{\mathrm{sh}^2 u + \sin^2 v}, \quad h_z = 1.$$

6. SPHEROID (ξ, η, φ)

a)

$$x = a\,\mathrm{sh}\,\xi\,\sin\eta\,\cos\varphi, \quad y = a\,\mathrm{sh}\,\xi\,\sin\eta\,\sin\varphi,$$
$$z = a\,\mathrm{ch}\,\xi\,\cos\eta$$
$$\text{gde je} \quad \xi \geq 0, \quad 0 \leq \eta \leq \pi, \quad 0 \leq \varphi < 2\pi$$
$$\text{i} \quad h_\xi = h_\eta = a\sqrt{\mathrm{sh}^2\xi + \sin^2\eta}, \quad h_\varphi = a\,\mathrm{sh}\xi\,\sin\eta.$$

b)

$$x = a\,\mathrm{ch}\,\xi\,\cos\eta\,\cos\varphi, \quad y = a\,\mathrm{ch}\,\xi\,\cos\eta\,\sin\varphi,$$
$$z = a\,\mathrm{sh}\,\xi\,\sin\eta$$
$$\text{where} \quad \xi \geq 0, \quad -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}, \quad 0 \leq \varphi < 2\pi$$
$$\text{i} \quad h_\xi = h_\eta = a\sqrt{\mathrm{sh}^2\xi + \sin^2\eta}, \quad h_\varphi = a\,\mathrm{ch}\xi\,\cos\eta.$$

7. ELLIPSOIDAL (α, β, φ)

$$x = c\sin\beta\cos\varphi, \quad y = c\sin\alpha\sin\beta\sin\varphi, \quad z = c\,\mathrm{ch}\,\alpha\cos\beta$$
$$\text{where}$$
$$0 \leq \alpha < \infty, \quad 0 \leq \beta \leq \pi, \quad -\pi < \varphi \leq \pi$$
$$\text{and Lamé coefficients are}$$
$$h_\alpha = h_\beta = c\sqrt{\mathrm{sh}^2 + \sin^2\beta}, \quad h_\varphi = c\,\mathrm{sh}\alpha\,\sin\beta.$$

8. ELLIPSOID (λ, μ, ν)

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1 \qquad \lambda < c^2 < b^2 < a^2$$
$$\frac{x^2}{a^2-\mu} + \frac{y^2}{b^2-\mu} + \frac{z^2}{c^2-\mu} = 1 \qquad c^2 < \mu < b^2 < a^2$$
$$\frac{x^2}{a^2-\nu} + \frac{y^2}{b^2-\nu} + \frac{z^2}{c^2-\nu} = 1 \qquad c^2 < b^2 < \nu < a^2,$$

where

$$h_\lambda = \frac{1}{2}\sqrt{\frac{(\mu-\lambda)(\nu-\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}$$
$$h_\mu = \frac{1}{2}\sqrt{\frac{(\nu-\mu)(\lambda-\mu)}{(a^2-\mu)(b^2-\mu)(c^2-\mu)}}$$
$$h_\nu = \frac{1}{2}\sqrt{\frac{(\lambda-\nu)(\mu-\nu)}{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}.$$

9. BIPOLAR – planar (u, v, z)

$$x^2 + (y - a\cot u)^2 = a^2 \csc^2 u,$$
$$(x - a\coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v, \quad z = z$$
$$\text{ili} \quad x = \frac{a\,\mathrm{sh}\, v}{\mathrm{ch}\, v - \cos u}, \quad y = \frac{a \sin u}{\mathrm{ch}\, v - \cos u}, \quad z = z$$
$$\text{gde je} \quad 0 \le u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty.$$

For Lamé coefficients we obtain

$$h_u = h_v = \frac{a}{\mathrm{ch}\, v - \cos u}, \quad h_z = 1.$$

10. BIPOLAR – bispherical (u, v, φ)

$$x = \frac{c \sin u \cos\varphi}{\mathrm{ch} v - \cos u}, \quad y = \frac{c \sin u \sin\varphi}{\mathrm{ch} v - \cos u}, \quad z = \frac{c\,\mathrm{sh} v}{\mathrm{ch} v - \cos u}$$
$$h_u = h_v = \frac{c}{\mathrm{ch} v - \cos u}, \quad h_\varphi = \frac{c \sin u}{\mathrm{ch} v - \cos u}.$$

11. ELLIPTICAL (λ, μ, z)

$$x = c\lambda\,\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1-\mu^2)}, \quad z = z$$

Lamé (metric) coefficients are

$$h_\lambda = c\sqrt{\frac{\lambda^2-\mu^2}{\lambda^2-1}}, \quad h_\mu = c\sqrt{\frac{\lambda^2-\mu^2}{1-\mu^2}}, \quad h_z = 1.$$

12. TOROIDAL (α, β, φ)

$$x = \frac{c\,\mathrm{sh}\alpha\cos\varphi}{\mathrm{ch}\alpha - \cos\beta}, \quad y = \frac{c\,\mathrm{sh}\alpha\sin\varphi}{\mathrm{ch}\alpha - \cos\beta}, \quad z = \frac{c\sin\beta}{\mathrm{ch}\alpha - \cos\beta}$$

gde je: $0 \le \alpha < \infty, \quad -\pi < \beta \le \pi, \quad -\pi < \varphi \le \pi$

while Lamé coefficients are

$$h_\alpha = h_\beta = \frac{c}{\mathrm{ch}\alpha - \cos\beta}, \quad h_\varphi = \frac{c\,\mathrm{sh}\alpha}{\mathrm{ch}\alpha - \cos\beta}.$$

14. SPHEROIDAL – a (λ, μ, φ)

a)

$$x = c\lambda\mu, \quad y = c\sqrt{(\lambda^2-1)(1-\mu^2)}\cos\varphi, \quad z = c\sqrt{(\lambda^2-1)(1-\mu^2)}\sin\varphi,$$

gde je: $\lambda \le 1, \quad -1 \le \mu \le 1, \quad 0 \le \varphi \le 2\pi$

where Lamé coefficients are

$$h_\lambda = c\sqrt{\frac{\lambda^2-\mu^2}{\lambda^2-1}}, \quad h_\mu = c\sqrt{\frac{\lambda^2-\mu^2}{1-\mu^2}}, \quad h_\varphi = c\sqrt{(\lambda^2-1)(1-\mu^2)}.$$

b)

$$x = c\lambda\mu\sin\varphi, \quad y = c\sqrt{(\lambda^2-1)(1-\mu^2)}, \quad z = c\lambda\mu\cos\varphi,$$

where Lamé coefficients are $\mu^2\lambda^2 - 1, \quad h_\mu = c\sqrt{\frac{\lambda^2-\mu^2}{1-\mu^2}}, \quad h_\varphi = c\lambda\mu.$

4.6 Examples

4.6.1 Gradient

Problem 33

Find the gradient for the following functions

a) $\phi = r$,

b) $\phi = \ln r$,

c) $\phi = \dfrac{1}{r}$,

where $\mathbf{r}$ is the position vector, and r its magnitude.

R For the physical interpretation see Chapter 4.3.

Solution

a) The position vector, expressed with respect to the Cartesian coordinate system, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, has the magnitude $r = \sqrt{x^2 + y^2 + z^2}$, and thus the scalar function ϕ, expressed with respect to the Cartesian coordinate system, has the following form

$$\phi = r = \sqrt{x^2 + y^2 + z^2}.$$

Its gradient is

$$\nabla\phi = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}\mathbf{i} + \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y}\mathbf{j} + \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z}\mathbf{k} \quad \Rightarrow$$

$$\nabla\phi = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k} =$$

$$= \frac{\mathbf{r}}{r} = \mathbf{r}_0.$$

b) The scalar function ϕ, expressed with respect to the Cartesian coordinate system, has the following form

$$\phi = \ln r = \frac{1}{2}\ln(x^2 + y^2 + z^2),$$

and thus its gradient is

$$\nabla\phi = \frac{1}{2}\nabla\ln(x^2+y^2+z^2) \quad \Rightarrow$$

$$\nabla\phi = \frac{1}{2}\left[\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \right.$$

$$\left. + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right] \quad \Rightarrow$$

$$\nabla\phi = \frac{1}{2}\left[\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right] \quad \Rightarrow$$

$$\nabla\phi = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}.$$

c) As in the previous example, we first express $\frac{1}{r}$ with respect to the Cartesian coordinate system, and then compute the gradient

$$\nabla\phi = \nabla\frac{1}{r} = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right] \quad \Rightarrow$$

$$\nabla\phi = \mathbf{i}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}\right] + \mathbf{j}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}\right] + \mathbf{k}\left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}\right] \quad \Rightarrow$$

$$\nabla\phi = \mathbf{i}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x\right] + \mathbf{j}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2y\right] + \mathbf{k}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2z\right] \quad \Rightarrow$$

$$\nabla\phi = -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{\mathbf{r}}{r^3}.$$

Problem 34

Prove that $\nabla r^n = n\,r^{n-2}\,\mathbf{r}$.

Solution

$$\nabla r^n = \nabla\sqrt{(x^2+y^2+z^2)}^{\,n} = \nabla(x^2+y^2+z^2)^{n/2} \quad \Rightarrow$$

$$\nabla r^n = \mathbf{i}\left[\frac{n}{2}(x^2+y^2+z^2)^{(n/2)-1}\cdot 2x\right] + \mathbf{j}\left[\frac{n}{2}(x^2+y^2+z^2)^{(n/2)-1}\cdot 2y\right] + \mathbf{k}\left[\frac{n}{2}(x^2+y^2+z^2)^{(n/2)-1}\cdot 2z\right] \quad \Rightarrow$$

$$\nabla r^n = n(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})(x^2+y^2+z^2)^{(n/2)-1}$$

$$\nabla r^n = n\,r^{n-2}\,\mathbf{r}.$$

R Note that in both this and previous example the solution can be obtained in a simpler way. See example 39 on page 123.

Problem 35

Prove that grad f is a vector orthogonal to the surface given by the function $f(x,y,z) = c$, where c is a constant.

Solution

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of point $P(x,y,z)$ on this surface. Then $\mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} + \mathrm{d}z\mathbf{k}$ belongs to the tangent plane of the surface in point P. Given that $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = \mathrm{d}(c) = 0$ or

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot (\mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} + \mathrm{d}z\mathbf{k}) = 0,$$

that is, $\nabla f \cdot \mathrm{d}\mathbf{r} = 0$, it follows that these two vectors (grad f and $\mathrm{d}\mathbf{r}$) are mutually orthogonal.

Problem 36

Find the equation of the tangent to the plane $2xz^2 - 3xy - 4x = 7$ in point $A(1,-1,2)$.

Solution

The equation of the plane through point A is

$$(\mathbf{r} - \mathbf{r}_A) \cdot \mathbf{n} = 0,$$

where $\mathbf{n}$ is the unit vector of the orthogonal vector of that plane in point A.

Thus, in order to determine the equation of the plane we need the vector orthogonal to that plane. According to the previous example, vector

$$\nabla f = \nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4)\mathbf{i} - 3x\mathbf{j} + 4xz\mathbf{k},$$

is the vector in the direction of the normal to the observed plane. The unit vector of this vector ($\mathbf{n} = \nabla f / |\nabla f|$), in point $A(1,-1,2)$, is $\mathbf{n} = 7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$. The equation of the tangent plane, in this case, is

$$[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})] \cdot \frac{(7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k})}{\sqrt{122}} = 0,$$

that is

$$7(x-1) - 3(y+1) + 8(z-2) = 0,$$
$$7x - 3y + +8z - 26 = 0.$$

Problem 37

Let T(x,y,z) and T $(x+\Delta x, y+\Delta y, z+\Delta z)$ be temperature values in two close points $P(x,y,z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$.

a) Give a physical interpretation of the value

$$\frac{\Delta T}{\Delta s} = \frac{T(x+\Delta x, y+\Delta y, z+\Delta z) - T(x,y,z)}{\Delta s}$$

where Δs is the distance between points P and Q.

b) Give a physical interpretation of the value $\lim\limits_{\Delta s\to 0}\frac{\Delta T}{\Delta s}=\frac{\mathrm{d}T}{\mathrm{d}s}$.

c) Prove that $\frac{\mathrm{d}T}{\mathrm{d}s}=\nabla T\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$.

Solution

a) ΔT represents the temperature increment in the transition from point P to point Q, Δs the distance between these two points, whereas their quotient $\frac{\Delta T}{\Delta s}$ represents the mean value of temperature change per unit length in the direction of PQ.

b) As the increment is equal to $\Delta T=\frac{\partial T}{\partial x}\Delta x+\frac{\partial T}{\partial y}\Delta y+\frac{\partial T}{\partial z}\Delta z+$ higher order infinitesimal values $\Delta x,\Delta y,\Delta z$, it follows that

$$\lim_{\Delta s\to 0}\frac{\Delta T}{\Delta s}=\lim_{\Delta s\to 0}\left(\frac{\partial T}{\partial x}\frac{\Delta x}{\Delta s}+\frac{\partial T}{\partial y}\frac{\Delta y}{\Delta s}+\frac{\partial T}{\partial z}\frac{\Delta z}{\Delta s}\right)$$

and consequently

$$\frac{\mathrm{d}T}{\mathrm{d}s}=\frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}.$$

Thus, $\frac{\mathrm{d}T}{\mathrm{d}s}$ represents the temperature change in point P, in the direction of PQ.

c)

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}s}&=\frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}=\\&=\left(\frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{i}+\frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{j}+\frac{\mathrm{d}z}{\mathrm{d}s}\mathbf{k}\right)=\\&=\nabla T\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}.\end{aligned}$$

Problem 38

Find the increment of the function $\phi=x^2yz+4xz^2$ in point $A(1,-2,-1)$, in the direction of vector $\mathbf{v}=2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

Solution

As the gradient in an arbitrary point is

$$\nabla\phi=\nabla(x^2yz+4xz^2)=(2xyz+4z^2)\mathbf{i}+x^2z\mathbf{j}+(x^2y+8xz)\mathbf{k},$$

it follows that the gradient in point $A(1,-2,-1)$ is

$$\nabla\phi=8\mathbf{i}-\mathbf{j}-10\mathbf{k}.$$

The unit vector in the direction $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$ is

$$\mathbf{n}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+(-1)^2+(-2)^2}}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}.$$

Then the required increment in the direction of $\mathbf{v}$ (see p.84) is

$$\nabla\phi\cdot\mathbf{n}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\cdot\left(\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}\right)=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

R Note that the solution is positive, namely $\frac{d\phi}{ds}>0$, which means that ϕ is growing in the given direction.

Problem 39

Prove that the gradient of a complex function $\varphi(f)$ is given in the form

$$\nabla\varphi(f)=\frac{d\varphi}{df}\nabla f,$$

where $f=f(x,y,z)$.

Proof

$$\nabla\varphi=\frac{\partial\varphi}{\partial x}\mathbf{i}+\frac{\partial\varphi}{\partial y}\mathbf{j}+\frac{\partial\varphi}{\partial z}\mathbf{k}=\frac{d\varphi}{df}\frac{\partial f}{\partial x}\mathbf{i}+\frac{d\varphi}{df}\frac{\partial f}{\partial y}\mathbf{j}+\frac{d\varphi}{df}\frac{\partial f}{\partial z}\mathbf{k}=$$
$$=\frac{d\varphi}{df}\left(\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}\right)=\frac{d\varphi}{df}\nabla f.$$

Let us apply this formula to the function $\varphi=\frac{1}{r}$ (see example 33c).

$$\nabla\varphi=\frac{d\varphi}{dr}\nabla r=-\frac{1}{r^2}\nabla r.$$

Given that (see example 33a, p. 119),

$$\nabla r=\frac{\mathbf{r}}{r}=\mathbf{r}_0$$

we finally obtain

$$\nabla\varphi=-\frac{\mathbf{r}}{r^3}=-\frac{1}{r^2}\mathbf{r}_0,$$

which is the same result as in the aforementioned example.

Problem 40

A scalar function $\phi = x^2yz^3$ is given.

a) Find the direction, through point $A(2,1,-1)$, in which the function is growing the fastest.

b) Compute the maximum increment of the function.

Solution

a) Bearing in mind the statement on page 84, that "...the gradient determines the direction in which the scalar field changes the fastest", we need to find the gradient of the function ϕ in point $A(2,1,-1)$

$$\nabla\phi = \nabla(x^2yz^3) = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k} = \\ = -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}.$$

Thus the direction, which passes through point A, and in which the function s growing the fastest is

$$\nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}.$$

b) The maximum increment is

$$|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176} = 4\sqrt{11}.$$

Problem 41

Find the angle between the surfaces $\phi_1 \equiv x^2 + y^2 + z^2 = 9$ and $\phi_2 \equiv z = x^2 + y^2 - 3$, at point $A(2,-1,2)$.

Solution

The angle between two surfaces at a given point is the angle between their normals, i.e, their gradients, at that point. The gradient (normal) of the surface $x^2 + y^2 + z^2 = 9$ at point $(2,-1,2)$ is (see Example 39)

$$\nabla\phi_1 = \nabla(x^2 + y^2 + z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k},$$

and the gradient (normal) of the surface $z = x^2 + y^2 - 3$ at point $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2 + y^2 - z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The scalar product of these vectors is $(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$, and thus the cosine of the required angle is

$$\cos\theta = \frac{(4\mathbf{i}-2\mathbf{j}+4\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})}{|4\mathbf{i}-2\mathbf{j}+4\mathbf{k}||4\mathbf{i}-2\mathbf{j}-\mathbf{k}|} = \\ = \frac{16+4-4}{\sqrt{(4)^2+(-2)^2+(4)^2}\sqrt{(4)^2+(-2)^2+(-1)^2}} = \\ = \frac{16}{6\sqrt{21}}.$$

From here we obtain $\theta = \arccos\dfrac{8}{3\sqrt{21}} = \arccos 0.5819 = 54.41^\circ$.

Problem 42

By taking samples from an open pit, determining their densities and interpolating the data, a density field $\rho = x^2 y$ was obtained. Find

a) gradient of this field at point $A(3,2)$,
b) density increment at point A in the direction defined by point $P(1,2)$,
c) density increment at point A in the direction AB, where point B is $B(1,3)$,
d) density increment at point A in the direction defined by point $Q(-4,3)$.

Solution

a) $\text{grad}\rho = 12\mathbf{i}+9\mathbf{j}$,
b) $\Delta_{u_P}\rho = 30\sqrt{5}$, density increases in this direction,
c) $\Delta_{AB}\rho = -13\sqrt{5}$, density decreases in this direction,
d) $\Delta_{u_Q}\rho = 0$, density is constant in this direction, i.e. in this direction the field is homogenous.

Problem 43

Let R be the distance between a fixed point $A(a,b,c)$ and an arbitrary point $P(x,y,z)$ $(R = \overline{AP})$. Prove that ∇R is the unit vector of vector $\overrightarrow{AP}$.

Solution

If $\mathbf{r}_A$ and $\mathbf{r}_P$ are position vectors $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ of points A and P respectively, then

$$\mathbf{R} = \mathbf{r}_P - \mathbf{r}_A = (x-a)\mathbf{i}+(y-b)\mathbf{j}+(z-c)\mathbf{k}$$

and the magnitude of this vector (the distance $\overline{AP}$) is

$$R = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

From here we obtain

$$\nabla R = \nabla\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = \\ = \frac{(x-a)\mathbf{i}+(y-b)\mathbf{j}+(z-c)\mathbf{k}}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} = \frac{\mathbf{R}}{R},$$

which is the unit vector of vector $\mathbf{R} = \overrightarrow{AP}$.

Problem 44

Let P be a point on the ellipse, and A and B be the foci of that ellipse. Show that the angle between AP and the tangent of the ellipse at point P is equal to the angle between BP and the tangent at the same point.

Solution

Let $\mathbf{R}_1 = \overrightarrow{AP}$ and $\mathbf{R}_2 = \overrightarrow{BP}$ be vectors connecting points A and B with point P, respectively, and $\mathbf{T}$ be the unit vector of tangent in point P (see Fig. 4.24).

From the definition of an ellipse it follows that the sum of the distances from the foci to the point P is a constant p, i.e. $R_1 + R_2 = p = \text{const}$.

As the unit vector of the normal to the ellipse is $\nabla(R_1+R_2)$, it follows that $[\nabla(R_1+R_2)]\cdot\mathbf{T} = 0$ or $\nabla R_2 \cdot \mathbf{T} = -\nabla R_1 \cdot \mathbf{T}$. Given that ∇R_1 and ∇R_2 are unit vectors of vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ (se Example 43, p. 125), it follows that the cosine of the angle between $\mathbf{R}_2$ and $\mathbf{T}$ is equal to the cosine of the angle between $\mathbf{R}_1$ and $-\mathbf{T}$. Consequently, these angles are equal.

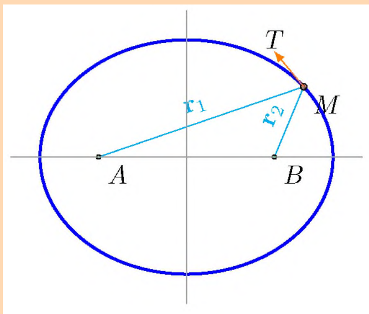


Figure 4.24: Ellipse.

R Note that the physical meaning of this result can be found in optics. Namely, the light emitted from point A is reflected by an elliptical mirror and passes through point B and vice versa.

4.6.2 Divergence

Problem 45

Find div$\mathbf{A}$ for the vector function $\mathbf{A} = x^2z\mathbf{i} + 2y^3z^2\mathbf{j} - xy^2z\mathbf{k}$ at point (1,1,-1).

Solution

$$\text{div}\mathbf{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(2y^2 z^3) - \frac{\partial}{\partial z}(xy^2 z) =$$
$$= 2xz + 6y^2 z^2 - xy^2 = -2 + 6 - 1 = 3.$$

Problem 46

For $\Phi = 3x^2 z - y^2 z^3 + 4x^3 y + 2x - 3y - 5$ find $\nabla^2 \Phi$.

Solution

$$\nabla^2 \Phi = 6z + 24xy - 2z^3 - 6y^2 z.$$

Problem 47

For $\mathbf{A} = (3x^2 y - z)\mathbf{i} + (xz^3 + y^4)\mathbf{j} - 2x^3 z^2 \mathbf{k}$ find $\nabla(\nabla \cdot \mathbf{A})$ at point (2,-1,0).

Solution

$$\nabla(\nabla \cdot \mathbf{A}) = -6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}.$$

Problem 48

For $\mathbf{A} = 3xyz^2\mathbf{i} + 2xy^3\mathbf{j} - x^2 yz\mathbf{k}$ and $\Phi = 3x^2 - yz$ find
a) $\nabla \cdot \mathbf{A}$, b) $\mathbf{A} \cdot \nabla \Phi$, c) $\nabla \cdot (\Phi \mathbf{A})$, d) $\nabla \cdot (\nabla \Phi)$, at point (1,-1,1).

Solution

a) 4, b) -15, c) 1, d) 6.

Problem 49

Show that the excess volume of fluid that flows out, in a unit of time, from a unit volume of fluid space, represents the divergence of the velocity $\mathbf{v}$ of the fluid.

Solution

Let us observe the flow of the fluid.

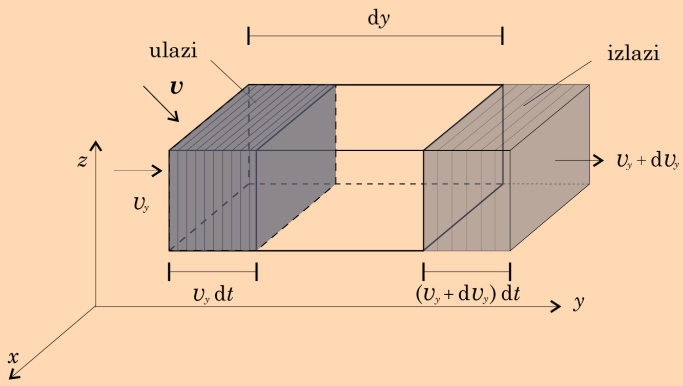


Figure 4.25: Flow of the fluid.

Let us calculate the difference between the flow of fluid entering the elemental volume $dV = dxdydz$ and the fluid leaving that volume, in the same time interval.

The amount of fluid entering this volume, in the direction of y-axis is

$$v_y dt dx dz.$$

Let us assume that changes of velocity v_y in directions of x and z are negligible, i.e. $\frac{\partial v_y}{\partial x} = 0$ i $\frac{\partial v_y}{\partial z} = 0$, due to small dimensions of the observed volume.

The amount of fluid that flows out, under the same assumption is

$$(v_y + dv_y)\, dt dx dz,$$

and thus the difference is

$$(v_y + dv_y)\, dt dx dz - v_y dt dx dz = dv_y dt dx dz.$$

This difference can be represented as follows

$$dv_y dt dx dz = \left(\frac{\partial v_y}{\partial y} dy + \frac{\partial v_y}{\partial x} dx + \frac{\partial v_y}{\partial z} dz\right) dt dx dz = \frac{\partial v_y}{\partial y} dt dV.$$

It can analogously be shown that changes in the directions of x and z are

$$dv_x dt dy dz, \quad dv_z dt dx dy,$$

respectively. The total change is equal to the sum of these changes, i.e.

$$\begin{aligned} d(dV) &= \frac{\partial v_x}{\partial x} dV dt + \frac{\partial v_y}{\partial y} dV dt + \frac{\partial v_z}{\partial z} dV dt = \\ &= (\operatorname{div}\mathbf{v}) dV dt \quad \Rightarrow \\ \operatorname{div}\mathbf{v} &= \frac{d(dV)}{dt dV}. \end{aligned}$$

Thus, the divergence of the fluid velocity $\mathbf{v}$ represents the change in fluid volume in unit time per unit volume of fluid space.

Problem 50

A scalar function $\phi = 2x^3y^2z^4$ is given.

a) Calculate $\nabla \cdot \nabla\phi$.

b) Show on this example that $\nabla \cdot \nabla\phi = \nabla^2\phi = \triangle\phi$, where $\nabla^2 \equiv \triangle \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the so-called Laplace operator (expressed with respect to Cartesian coordinates).

Solution

a) Let us start from the definition of the gradient

$$\begin{aligned}\nabla\phi =& \mathbf{i}\frac{\partial}{\partial x}\left(2x^3y^2z^4\right) + \mathbf{j}\frac{\partial}{\partial y}\left(2x^3y^2z^4\right) + \mathbf{k}\frac{\partial}{\partial z}\left(2x^3y^2z^4\right) = \\ =& 6x^2y^2z^4\mathbf{i} + 4x^3yz^4\mathbf{j} + 8x^3y^2z^3\mathbf{k}.\end{aligned}$$

Let us now find the divergence of this vector, i.e.

$$\begin{aligned}\nabla \cdot \nabla\phi =& \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(6x^2y^2z^4\mathbf{i} + 4x^3yz^4\mathbf{j} + 8x^3y^2z^3\mathbf{k}\right) = \\ =& \frac{\partial}{\partial x}\left(6x^2y^2z^4\right) + \frac{\partial}{\partial y}\left(4x^3yz^4\right) + \frac{\partial}{\partial z}\left(8x^3y^2z^3\right) = \\ =& 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2.\end{aligned}$$

b) Given that

$$\triangle\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2,$$

by comparison with the result under a), we can see on this example that

$$\nabla \cdot \nabla\phi = \triangle\phi = \nabla^2\phi.$$

Problem 51

Prove the following properties of divergence

a) $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

b) $\nabla \cdot (\phi\mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi\nabla \cdot \mathbf{A}$

Solution

a) Given that (with respect to the Cartesian coordinate system)

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

and

$$\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k},$$

it follows that

$$\begin{aligned}\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot[(A_1+B_1)\mathbf{i}+(A_2+B_2)\mathbf{j}+(A_3+B_3)\mathbf{k}] =\\ &=\frac{\partial(A_1+B_1)}{\partial x}+\frac{\partial(A_2+B_2)}{\partial y}+\frac{\partial(A_3+B_3)}{\partial z}=\\ &=\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}=\\ &=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}.\end{aligned}$$

b)

$$\begin{aligned}\nabla\cdot(\phi\mathbf{A}) &=\nabla\cdot(\phi A_1\mathbf{i}+\phi A_2\mathbf{j}+\phi A_3\mathbf{k})=\\ &=\frac{\partial(\phi A_1)}{\partial x}+\frac{\partial(\phi A_2)}{\partial y}+\frac{\partial(\phi A_3)}{\partial z}=\\ &=\frac{\partial\phi}{\partial x}A_1+\frac{\partial\phi}{\partial y}A_2+\frac{\partial\phi}{\partial z}A_3+\frac{\partial A_1}{\partial x}\phi+\frac{\partial A_2}{\partial y}\phi+\frac{\partial A_3}{\partial z}\phi=\\ &=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})+\\ &\qquad+\phi\cdot\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)=\\ &=(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A}).\end{aligned}$$

Problem 52

Show that $\nabla^2\left(\frac{1}{r}\right)=\triangle\left(\frac{1}{r}\right)=0$, where $r=\sqrt{x^2+y^2+z^2}$.

Solution

$$\begin{aligned}\nabla^2\left(\frac{1}{r}\right) &=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\sqrt{x^2+y^2+z^2}\right)^{-1}=\\ &=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}.\end{aligned}$$

Given that

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}=-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}},$$

it follows that

$$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-\frac{1}{2}}=\frac{\partial}{\partial x}(-x(x^2+y^2+z^2)^{-\frac{3}{2}})=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}. \qquad (4.153)$$

Analogously we obtain

$$\frac{\partial^2(x^2+y^2+z^2)^{-\frac{1}{2}}}{\partial y^2}=\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \qquad (4.154)$$

$$\frac{\partial^2 (x^2+y^2+z^2)^{-\frac{1}{2}}}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}. \tag{4.155}$$

By adding equations (4.153), (4.154) and (4.155) we now obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=0. \tag{4.156}$$

Recall that the equation $\nabla^2\phi = 0$ is called the Laplace equation, and its solution is called the harmonic function. Therefore, the solution of equation (4.156) $\phi = 1/r$ is a harmonic function[18].

The problem in this Example can be solved in yet another way.

Given that

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\nabla\left(\frac{1}{r}\right),$$

and using the result from Example 33, on p. 119, we obtain

$$\nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}.$$

According to Example 53, on p. 131, it follows that

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)=0,$$

which proves that $\varphi = 1/r$ is a harmonic function.

Problem 53

Prove that $\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)=0$.

Proof

If we introduce the notation $\phi = r^{-3}$ and $\mathbf{A} = \mathbf{r}$ the problem is reduced to the application of the previously proved property of divergence, where Example 34 on p.120 is used, for $n = -3$,:

$$\begin{aligned}\nabla\cdot(r^{-3}\mathbf{r}) &= \left(\nabla r^{-3}\right)\cdot\mathbf{r}+(r^{-3}\nabla\cdot\mathbf{r})\\ &= -3r^{-5}\mathbf{r}\cdot\mathbf{r}+3r^{-3}=0.\end{aligned}$$

Problem 54

Prove that $\nabla\cdot(U\nabla V - V\nabla U) = U\nabla^2 V - V\nabla^2 U$.

[18] Note that $\phi = 1/r$ is a harmonic function in space E_3, whereas $y = \ln r$ is a harmonic function in space E_2.

Solution

If we refer to Example 51b on p.129 and introduce the notation $\phi = U$, that is, $\mathbf{A} = \nabla V$, then

$$\nabla \cdot (U\nabla V) = (\nabla U)\cdot(\nabla V) + U(\nabla\cdot\nabla V) = (\nabla U)\cdot(\nabla V) + U\nabla^2 V, \tag{4.157}$$
$$\nabla \cdot (V\nabla U) = \qquad = (\nabla V)\cdot(\nabla U) + V\nabla^2 U. \tag{4.158}$$

The relation (4.158) was obtained from (4.157) by replacing U by V and V by U. Subtracting the left and right sides and using the divergence property (the divergence of the difference is the difference of the divergence), we obtain

$$\begin{aligned}\nabla\cdot(U\nabla V) - \nabla\cdot(V\nabla U) =&\\ =&\nabla\cdot(U\nabla V - V\nabla U) =\\ =&(\nabla U)\cdot(\nabla V) + U\nabla^2 V - [(\nabla V)\cdot(\nabla U) + V\nabla^2 U] =\\ =&U\nabla^2 V - V\nabla^2 U.\end{aligned}$$

Problem 55

Find the constant a for which the vector field

$$\mathbf{V} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k},$$

is solenoid.

Solution

The condition for a vector field to be solenoid is $\nabla\cdot\mathbf{V} = 0$ (see p. 92). Given that

$$\nabla\cdot\mathbf{V} = \frac{\partial(x+3y)}{\partial x} + \frac{\partial(y-2z)}{\partial y} + \frac{\partial(x+az)}{\partial z} = 1+1+a,$$

this condition yields

$$\nabla\cdot\mathbf{V} = a+2 = 0,$$

that is, $a = -2$.

4.6.3 Rotor

Exercise 56

For $\mathbf{A} = x^2y\mathbf{i} - 2xz\mathbf{j} + 2yz\mathbf{k}$ find rot rot$\mathbf{A}$.

Solution

$$\text{rot}\,\text{rot}\mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = (2x+2)\mathbf{j}.$$

Exercise 57

For $\mathbf{A} = 2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k}$ and $\Phi = 2x^2yz^3$ find $\mathbf{A} \times \nabla\Phi$.

Solution

$\mathbf{A} \times \nabla\Phi = -(6x^4y^2z^2 + 2x^3z^5)\mathbf{i} + (4x^2yz^5 - 12x^2y^2z^3)\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\mathbf{k}.$

Exercise 58

Show that the field $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not rotational (rot$\mathbf{A} = 0$), and that the field $\mathbf{B} = xyz^2\mathbf{A}$ is rotational.

Exercise 59

For $\mathbf{A} = xz^2\mathbf{i} + 2y\mathbf{j} - 3xz\mathbf{k}$ and $\mathbf{B} = 3xz\mathbf{i} + 2yz\mathbf{j} - z^2\mathbf{k}$ find $\mathbf{A} \times (\nabla \times \mathbf{B})$ at point $(1,-1,2)$.

Solution

$\mathbf{A} \times (\nabla \times \mathbf{B}) = 18\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}.$

Exercise 60

Find rot$\mathbf{V}$ and div$\mathbf{V}$ for the vector field $\mathbf{V} = -\frac{\mathbf{r}}{r} = \frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2+y^2}}$.

Exercise 61

Show that rot rot$\mathbf{A} = -\Delta\mathbf{A} + \text{grad div}\mathbf{A}$.

Exercise 62

Determine the velocity rotor of an arbitrary point on a rigid body rotating around a fixed pole.

Solution

The velocity of a point on a body rotating around a fixed pole (which can be assumed to be the origin of the coordinate system, without loss of generality) is given by the relation

$$\mathbf{v} = \omega \times \rho = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \mathbf{i}\left(\omega_y z - \omega_z y\right) + \mathbf{j}\left(\omega_z x - \omega_x z\right) + \mathbf{k}\left(\omega_x y - \omega_y x\right),$$

where ω is the angular velocity, and ρ the position vector of the point on the body. Given that the angular velocity of a body is equal for all points on this body, it does not depend on the coordinates x, y and z, and it follows that

$$\begin{aligned}
&\frac{\partial}{\partial x}(\dot{y}) = \frac{\partial}{\partial x}\left(\omega_z x - \omega_x z\right) = \omega_z \quad &&\frac{\partial}{\partial y}(\dot{x}) = \frac{\partial}{\partial y}\left(\omega_y z - \omega_z y\right) = -\omega_z, \\
&\frac{\partial}{\partial z}(\dot{x}) = \frac{\partial}{\partial z}\left(\omega_y z - \omega_z y\right) = \omega_y \quad &&\frac{\partial}{\partial x}(\dot{z}) = \frac{\partial}{\partial x}\left(\omega_x y - \omega_y x\right) = -\omega_y, \\
&\frac{\partial}{\partial y}(\dot{z}) = \frac{\partial}{\partial y}\left(\omega_x y - \omega_y z\right) = \omega_x \quad &&\frac{\partial}{\partial z}(\dot{y}) = \frac{\partial}{\partial z}\left(\omega_z x - \omega_x z\right) = -\omega_x.
\end{aligned}$$

Thus, the velocity rotor $\mathbf{v}$, according to (4.46) is

$$\text{rot}\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \mathbf{i}\left(\frac{\partial \dot{z}}{\partial y} - \frac{\partial \dot{y}}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \dot{x}}{\partial z} - \frac{\partial \dot{z}}{\partial x}\right) + \mathbf{k}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y}\right).$$

From these relations we finally obtain

$$\text{rot}\mathbf{v} = \mathbf{i}[\omega_x - (-\omega_x)] + \mathbf{j}[\omega_y - (-\omega_y)] + \mathbf{k}[\omega_z - (-\omega_z)] = 2\omega.$$

Exercise 63

If $\mathbf{V} = xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$, find $\nabla \times \mathbf{V}$ at point $A(1, -1, 1)$.

Solution

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}) =$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} =$$

$$= \left[\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(xz^3)}{\partial z} - \frac{\partial(2yz^4)}{\partial x}\right]\mathbf{j} + \left[\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y}\right]\mathbf{k} =$$

$$= (2z^4 + 2xz)\mathbf{i} + (3xz^2)\mathbf{j} - (4xyz)\mathbf{k}.$$

For point $A(1,-1,1)$ we obtain $\nabla \times \mathbf{V}\Big|_A = 3\mathbf{j} + 4\mathbf{k}$.

Exercise 64

Observe the expression $\nabla \cdot (\mathbf{V} \times \mathbf{W})$.

a) Prove that

$$\nabla \cdot (\mathbf{V} \times \mathbf{W}) = (\nabla \times \mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot (\nabla \times \mathbf{W}).$$

b) Find the value for the expression if $\mathbf{W} = \mathbf{r}$ and $\nabla \times \mathbf{V} = 0$.

c) Prove that, if $\mathbf{V}$ and $\mathbf{W}$ are irrotational (potential) fields, then the field obtained as their vector product is solenoidal (rotational).

d) It follows from b) and c) that the vector field of a vector product of a position vector and any irrotational vector field is a solenoidal field. Prove this statement.

e) The vector field of a vector product of a position vector and a conservative force is always a solenoidal field. Prove this statement.

Solution

a) Given that

$$\nabla \cdot (\mathbf{V} \times \mathbf{W}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = \tag{4.159}$$

$$= \frac{\partial}{\partial x}(V_yW_z - V_zW_y) + \frac{\partial}{\partial y}(V_zW_x - V_xW_z) + \frac{\partial}{\partial z}(V_xW_y - V_yW_x),$$

after differentiating and grouping, we obtain

$$W_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + W_y\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + W_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) - \\ -V_x\left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z}\right) - V_y\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) - V_z\left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y}\right). \tag{4.160}$$

Given that

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} =$$

$$= \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right),$$

and similarly for the vector field **W**, the relation (4.159), using (4.160) becomes

$$\nabla \cdot (\mathbf{V} \times \mathbf{W}) = (\nabla \times \mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot (\nabla \times \mathbf{W}), \qquad (4.161)$$

which was to be proved.

b) By replacing **W** with **r** in relation (4.161), we obtain

$$\nabla \cdot (\mathbf{V} \times \mathbf{r}) = (\nabla \times \mathbf{V}) \cdot \mathbf{r} - \mathbf{V} \cdot (\nabla \times \mathbf{r}). \qquad (4.162)$$

Given that, according to the initial assumption $\nabla \times \mathbf{V} = 0$, and according to (4.62) $\nabla \times \mathbf{r} = 0$ (see p. 94), it follows that

$$\nabla \cdot (\mathbf{V} \times \mathbf{r}) = \mathbf{r} \cdot (\nabla \times \mathbf{V}) = 0.$$

c) Based on the relation (4.161), we can conclude that the divergence of the vector product is equal to zero, because fields **V** and **W** are irrotational, i.e. their rotors are equal to zero.

d) Let **v** be an irrotational field i.e. $\nabla \times \mathbf{v} = 0$. The task is to examine the vector field **w**, defined by $\mathbf{w} = \mathbf{r} \times \mathbf{v}$.
Let us first compute the gradient of this field $\nabla \cdot \mathbf{w} = \nabla \cdot (\mathbf{r} \times \mathbf{v})$. Using the result under a), the relation (4.62) on p. 94, as well as the initial assumption $(\nabla \times \mathbf{v} = 0)$, we obtain

$$\nabla \cdot \mathbf{w} = \nabla \cdot (\mathbf{r} \times \mathbf{v}) = (\nabla \times \mathbf{r}) \cdot \mathbf{v} - \mathbf{r} \cdot (\nabla \times \mathbf{v}) = 0.$$

It can be proved that, in the general case, $\nabla \times \mathbf{w} \neq 0$. For proving this, use a property of the nabla operator (see p. 88).
Thus, the field **w** is a solenoidal (rotational) field.

e) Observe the vector field $\mathbf{M} = \mathbf{r} \times \mathbf{S}$, where **S** is the force, and **r** the position vector. As, according to the initial assumption, the force is conservative, i.e. $\nabla \times \mathbf{S} = 0$, it follows that

$$\nabla \cdot \mathbf{M} = \nabla \cdot (\mathbf{r} \times \mathbf{S}) = (\nabla \times \mathbf{r}) \cdot \mathbf{S} - \mathbf{r} \cdot (\nabla \times \mathbf{S}) = 0.$$

Exercise 65

Find $\mathrm{rot}(\mathbf{r} f(r))$, if $f(r)$ is a differentiable function.

Solution

$$\nabla\times(\mathbf{r}f(r)) = \nabla\times(xf(r)\mathbf{i}+yf(r)\mathbf{j}+zf(r)\mathbf{k}) =$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} =$$
$$= \left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)\mathbf{i} + \left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)\mathbf{j} + \left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Let us now compute the partial derivatives of the function f by x

$$\frac{\partial f}{\partial x} = \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\left(\frac{\partial r}{\partial x}\right) = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2}) = f^{'}(r)\frac{x}{\sqrt{x^2+y^2+z^2}} = f^{'}\frac{x}{r},$$

where $\dfrac{\mathrm{d}f}{\mathrm{d}r} \equiv f'$. In a similar way we can compute the remaining two partial derivatives

$$\frac{\partial f}{\partial y} = f^{'}\frac{y}{r} \quad \text{i} \quad \frac{\partial f}{\partial z} = f^{'}\frac{z}{r},$$

and thus finally obtain

$$\nabla\times(\mathbf{r}f(r)) =$$
$$\left(z\frac{y}{r}f^{'} - y\frac{z}{r}f^{'}\right)\mathbf{i} + \left(x\frac{z}{r}f^{'} - z\frac{x}{r}f^{'}\right)\mathbf{j} + \left(y\frac{x}{r}f^{'} - x\frac{y}{r}f^{'}\right)\mathbf{k} = \mathbf{0}.$$

Exercise 66

Prove that $\nabla\times(\nabla\times\mathbf{V}) = -\nabla^2\mathbf{V} + \nabla(\nabla\cdot\mathbf{V})$.

Solution

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} =$$
$$= \nabla\times\left[\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}\right] =$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V_3}{\partial y} - \dfrac{\partial V_2}{\partial z} & \dfrac{\partial V_1}{\partial z} - \dfrac{\partial V_3}{\partial x} & \dfrac{\partial V_2}{\partial x} - \dfrac{\partial V_1}{\partial y} \end{vmatrix} =$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right]\mathbf{i} +$$
$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]\mathbf{j} +$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\right]\mathbf{k} =$$

$$= \left(-\frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 V_2}{\partial z^2} - \frac{\partial^2 V_2}{\partial x^2}\right)\mathbf{j} + \left(-\frac{\partial^2 V_3}{\partial x^2} - \frac{\partial^2 V_3}{\partial y^2}\right)\mathbf{k} +$$
$$+ \left(\frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x}\right)\mathbf{i} + \left(\frac{\partial^2 V_3}{\partial z \partial y} + \frac{\partial^2 V_1}{\partial x \partial y}\right)\mathbf{j} + \left(\frac{\partial^2 V_1}{\partial x \partial z} + \frac{\partial^2 V_2}{\partial y \partial z}\right)\mathbf{k} =$$

$$= \left(-\frac{\partial^2 V_1}{\partial x^2} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 V_2}{\partial x^2} - \frac{\partial^2 V_2}{\partial y^2} - \frac{\partial^2 V_2}{\partial z^2}\right)\mathbf{j} +$$
$$+ \left(-\frac{\partial^2 V_3}{\partial x^2} - \frac{\partial^2 V_3}{\partial y^2} - \frac{\partial^2 V_3}{\partial z^2}\right)\mathbf{k} +$$
$$+ \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x \partial y} + \frac{\partial^2 V_3}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 V_1}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_3}{\partial y \partial z}\right)\mathbf{j} +$$
$$+ \left(\frac{\partial^2 V_1}{\partial z \partial x} + \frac{\partial^2 V_2}{\partial z \partial y} + \frac{\partial^2 V_3}{\partial z^2}\right)\mathbf{k} =$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) +$$
$$+ \frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)\mathbf{j} +$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)\mathbf{k} =$$
$$= -\nabla^2 \mathbf{V} + \nabla\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) =$$
$$= -\nabla^2 \mathbf{V} + \nabla(\nabla \cdot \mathbf{V})$$

Exercise 67

The velocity of an arbitrary point of a rigid body rotating around a fixed point is given by the expression $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Prove that $\boldsymbol{\omega} = \frac{1}{2}\operatorname{rot}\mathbf{v}$.

Solution

Note that when a rigid body rotates the angular velocity does not depend on the position of the point in the body, i.e. $\frac{\partial \omega_i}{\partial x} = \frac{\partial \omega_i}{\partial y} = \frac{\partial \omega_i}{\partial z} = 0$ $(i = 1, 2, 3)$, and thus it follows that

$$\nabla\times\mathbf{v}=\nabla\times(\omega\times\mathbf{r})=\nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}=$$

$$=\nabla\times[(\omega_2 z-\omega_3 y)\,\mathbf{i}+(\omega_3 x-\omega_1 z)\,\mathbf{j}+(\omega_1 y-\omega_2 x)\,\mathbf{k}]=$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x\end{vmatrix}=$$

$$=2\,(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k})=2\omega.$$

Exercise 68

Let

$$\nabla\cdot\mathbf{E}=0,\qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{4.163}$$

Prove that $\mathbf{E}$ and $\mathbf{B}$ satisfy the equation $\nabla^2\mathbf{u}=a^2\dfrac{\partial^2\mathbf{u}}{\partial t^2}$, where $\mathbf{E}$ is the strength of the electric field, $\mathbf{B}$ is the magnetic induction, and a a constant.

Solution

Observe the expression $\nabla\times(\nabla\times\mathbf{E})$, which can be expressed in the form (see Example 66, p.137)

$$\nabla\times(\nabla\times\mathbf{E})=-\nabla^2\mathbf{E}+\nabla(\nabla\cdot\mathbf{E}).$$

Using the initial assumption (4.163) ($\nabla\cdot\mathbf{E}=0\wedge\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$) we obtain

$$\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)=-\nabla^2\mathbf{E},$$

that is

$$-\frac{\partial}{\partial t}(\nabla\times\mathbf{B})=-\nabla^2\mathbf{E}.$$

If we now use the condition $\nabla\times\mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}$, we finally obtain for the field $\mathbf{E}$

$$\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\nabla^2\mathbf{E}. \tag{4.164}$$

Similarly, for the field $\mathbf{B}$ we obtain

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=\nabla^2\mathbf{B}. \tag{4.165}$$

Thus the equations (4.164) and (4.165) can be represented in the form

$$\nabla^2 \mathbf{u} = a^2 \frac{\partial^2 \mathbf{u}}{\partial t^2},$$

which was to be proved[19].

Exercise 69

Let the vector field $\mathbf{F}$ be described by the relation $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$. Compute

a) $\nabla \times \mathbf{F}$.
b)

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

along an arbitrary closed path. Explain the results, if $\mathbf{F}$ represents the force.

Solution

a) Let us start with the definition of rotor

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix},$$

from where we obtain rot$\mathbf{F} = \mathbf{0}$ in any area except in point (0,0). Thus, the vector field $\mathbf{F}$ is laminar. If $\mathbf{F}$ represents the force, then the field of the force is potential, and the force is conservative.

b) Observe the integral along a closed line

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}.$$

Due to the shape of the line, it is convenient to switch to the polar coordinate system, which is connected to the Cartesian coordinate system by relations

$$x = \rho \cos\varphi,$$
$$y = \rho \sin\varphi,$$

where (ρ, φ) are polar coordinates. By differentiating we obtain

$$\mathrm{d}x = -\rho \sin\varphi \,\mathrm{d}\varphi + \mathrm{d}\rho \cos\varphi$$
$$\mathrm{d}y = \rho \cos\varphi \,\mathrm{d}\varphi + \mathrm{d}\rho \sin\varphi.$$

The relevant value, expressed in the two coordinate systems, is now

$$\frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2} = \mathrm{d}\varphi = \mathrm{d}\left(\arctan \frac{y}{x}\right).$$

[19] Note that these equations are known as Maxwell's equations for the electromagnetic field.

Observe the following two cases

1) when the origin of the coordinate system is inside the closed curved line and
2) when the origin of the coordinate system is outside the closed curved line, see Fig. 4.26.

First case. Observe the closed curved line *ABCDA* (see Figure) which encircles the origin of the coordinate system. When moving along the curve, the top of the position vector starts at point A ($\mathbf{r} = \mathbf{r}_A$, $\varphi_0 = 0$), moves along the curve and returns to the starting point (the curve is closed !!!). Thus, the angle was increased by 2π ($0 \le \varphi \le 2\pi$), and the line integral is equal to

$$\int_0^{2\pi} \mathrm{d}\varphi = 2\pi.$$

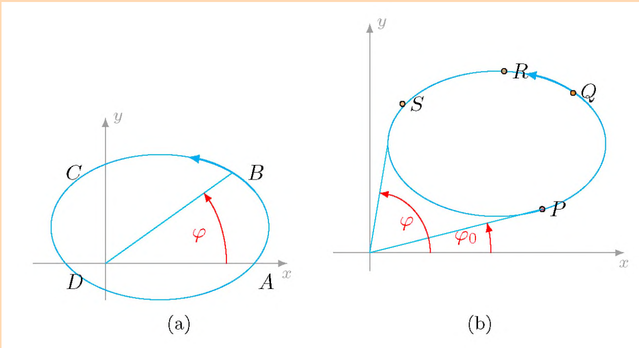


Figure 4.26: Origin of the coordinate system: (a) inside, (b) outside the closed curved line.

Second case. For the closed curve *PQRSP* (see Fig. 4.26) that does not encircle the origin of the coordinate system, the angle changes from $\varphi = \varphi_0$ in P to $\varphi = \varphi_0$ after completing the full circle. The line integral is equal to

$$\int_{\varphi_0}^{\varphi_0} \mathrm{d}\varphi = 0.$$

4.6.4 Mixed problems

Exercise 70

The following vector is given

$$\mathbf{v} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}.$$

a) Determine the constants a, b, c so that the vector field is potential.
b) Find a scalar function ϕ, whose gradient is equal to vector $\mathbf{v}$, for the values of constants determined under a).

Solution

a) Let us start from the definition of the rotor

$$\text{rot}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix},$$

where $v_x = x + 2y + az$, $v_y = bx - 3y - z$, $v_z = 4x + cy + 2z$. It follows that

$$\text{rot}\mathbf{v} = (c+1)\mathbf{i} + (a-4)\mathbf{j} + (b+2)\mathbf{k}.$$

According to the condition of this example the vector field is potential, namely rot**v**=0. This condition is fulfilled if $a = 4, b = 2, c = -1$, and it follows that

$$\mathbf{v} = (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - y + 2z)\mathbf{k}.$$

b) Given that

$$\mathbf{v} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k},$$

it follows, according to a), that

$$\frac{\partial\phi}{\partial x} = x + 2y + 4z, \tag{4.166}$$

$$\frac{\partial\phi}{\partial y} = 2x - 3y - z, \tag{4.167}$$

$$\frac{\partial\phi}{\partial z} = 4x - y + 2z. \tag{4.168}$$

As, in this case, $rot\mathbf{v} = \text{rotgrad}\varphi = 0$, this system of equations is integrable. By integrating the relation (4.166) over x, assuming that y and z are constant, we obtain

$$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z), \tag{4.169}$$

where $f(y,z)$ is a function of y and z.
By differentiating the equation (4.169) by y and equating it with the equation (4.167) we obtain

$$\frac{\partial\phi}{\partial y} = 2x + \frac{\partial f(y,z)}{\partial y} = 2x - 3y - z. \tag{4.170}$$

It follows that

$$\frac{\partial f(y,z)}{\partial y} = -3y - z. \tag{4.171}$$

By solving the equation (4.171), assuming that z is constant, we obtain

$$f(y,z) = -\frac{3y^2}{2} - yz + g(z). \tag{4.172}$$

Substituting the equation (4.172) into the equation (4.169) we obtain the following expression for the required function

$$\phi = \frac{x^2}{2} + 2xy + 4xz + -\frac{3y^2}{2} - yz + g(z) \tag{4.173}$$

Differentiating once again, this time equation (4.173) by z, we obtain

$$\frac{\partial \phi}{\partial z} = 4x - y + \frac{\partial g(z)}{\partial z}. \tag{4.174}$$

By equating the right hand side of equation (4.174) with the right hand side of equation (4.168), we obtain the unknown function $g(z)$

$$4x - y + \frac{\partial g(z)}{\partial z} = 4x - y + 2z, \tag{4.175}$$

$$\frac{\partial g(z)}{\partial z} = 2z, \tag{4.176}$$

$$g(z) = z^2 + \text{const.} \tag{4.177}$$

By substituting equation (4.177) into equation (4.173) we obtain the required scalar function

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{const.}$$

Exercise 71

Show that if the scalar function $\phi(x,y,z)$ is a solution of the Laplace equation, then the vector field $\nabla\phi$ is solenoid and potential.

Solution

According to the condition of the example the function ϕ is a solution of the Laplace equation $\nabla^2\phi = 0$, that is, $\nabla \cdot (\nabla\phi) = 0$. It follows from here that $\nabla\phi$ is a solenoid field. On the other hand, it is always true that $\nabla \times (\nabla\phi) = 0$, and thus the field $\nabla\phi$ is also potential.

Exercise 72

How can the concept of gradient be extended to vector functions?

Solution

Express the vector $\mathbf{B}$, for example, with respect to Cartesian coordinates $\mathbf{B} =$

$B_1 \cdot \mathbf{i} + B_2 \cdot \mathbf{j} + B_3 \cdot \mathbf{k}$. It follows that grad$\mathbf{B}$

$$\begin{aligned}\nabla \mathbf{B} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial x}\mathbf{j} + \frac{\partial}{\partial x}\mathbf{k}\right) \otimes (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) = \\ &= \left(\frac{\partial B_1}{\partial x}\mathbf{i}\otimes\mathbf{i} + \frac{\partial B_2}{\partial x}\mathbf{i}\otimes\mathbf{j} + \frac{\partial B_3}{\partial x}\mathbf{i}\otimes\mathbf{k}\right) + \\ &+ \left(\frac{\partial B_1}{\partial x}\mathbf{j}\otimes\mathbf{i} + \frac{\partial B_2}{\partial x}\mathbf{j}\otimes\mathbf{j} + \frac{\partial B_3}{\partial x}\mathbf{j}\otimes\mathbf{k}\right) + \\ &+ \left(\frac{\partial B_1}{\partial x}\mathbf{k}\otimes\mathbf{i} + \frac{\partial B_2}{\partial x}\mathbf{k}\otimes\mathbf{j} + \frac{\partial B_3}{\partial x}\mathbf{k}\otimes\mathbf{k}\right),\end{aligned}$$

where $\mathbf{i}\otimes\mathbf{i}$, $\mathbf{i}\otimes\mathbf{j}, \ldots, \mathbf{k}\otimes\mathbf{k}$ represent the so called unit dyads[20].

The value represented in the form

$$\begin{aligned}&a_{11}\mathbf{i}\otimes\mathbf{i} + a_{12}\mathbf{i}\otimes\mathbf{j} + a_{13}\mathbf{i}\otimes\mathbf{k} + \\ &+ a_{21}\mathbf{j}\otimes\mathbf{i} + a_{22}\mathbf{j}\otimes\mathbf{j} + a_{23}\mathbf{j}\otimes\mathbf{k} + \\ &+ a_{31}\mathbf{k}\otimes\mathbf{i} + a_{32}\mathbf{k}\otimes\mathbf{j} + a_{33}\mathbf{k}\otimes\mathbf{k}\end{aligned}$$

is a dyad and the constants a_{11}, a_{12}, a_{13}... are its components.

Exercise 73

Let the vector $\mathbf{A}$ be expressed in the following form $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, and dyad ϕ as $\phi = a_{11}\mathbf{i}\otimes\mathbf{i} + a_{12}\mathbf{i}\otimes\mathbf{j} + a_{13}\mathbf{i}\otimes\mathbf{k} + a_{21}\mathbf{j}\otimes\mathbf{i} + a_{22}\mathbf{j}\otimes\mathbf{j} + a_{23}\mathbf{j}\otimes\mathbf{k} + a_{31}\mathbf{k}\otimes\mathbf{i} + a_{32}\mathbf{k}\otimes\mathbf{j} + a_{33}\mathbf{k}\otimes\mathbf{k}$. Compute $\mathbf{A}\cdot\phi$.

Solution

Using the law of distribution we obtain

$$\mathbf{A}\cdot\phi = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\phi = A_1\mathbf{i}\cdot\phi + A_2\mathbf{j}\cdot\phi + A_3\mathbf{k}\cdot\phi.$$

Let us first compute $\mathbf{i}\cdot\phi$. This product is defined as the scalar product of the unit vector $\mathbf{i}$ with each component of the dyad ϕ. Typical elements are $\mathbf{i}\cdot a_{11}\mathbf{i}\otimes\mathbf{i}$, $\mathbf{i}\cdot a_{12}\mathbf{i}\otimes\mathbf{j}$, $\mathbf{i}\cdot a_{21}\mathbf{j}\otimes\mathbf{i}$, $\mathbf{i}\cdot a_{32}\mathbf{k}\otimes\mathbf{j}$, ... Taking this into account, we obtain

$$\begin{aligned}\mathbf{i}\cdot a_{11}\mathbf{i}\otimes\mathbf{i} &= a_{11}(\mathbf{i}\cdot\mathbf{i})\mathbf{i} = \mathbf{a_{11}i}, \\ \mathbf{i}\cdot a_{12}\mathbf{i}\otimes\mathbf{j} &= a_{12}(\mathbf{i}\cdot\mathbf{i})\mathbf{j} = \mathbf{a_{12}j}, \\ \mathbf{i}\cdot a_{21}\mathbf{j}\otimes\mathbf{i} &= a_{21}(\mathbf{i}\cdot\mathbf{j})\mathbf{i} = \mathbf{0}, \\ \mathbf{i}\cdot a_{32}\mathbf{k}\otimes\mathbf{j} &= a_{32}(\mathbf{i}\cdot\mathbf{k})\mathbf{j} = \mathbf{0}.\end{aligned}$$

[20] An ordered pair of vectors $(\mathbf{a}, \mathbf{b})$, denoted by $\mathbf{a}\otimes\mathbf{b}$ or $\mathbf{ab}$ (the older way of denoting), is called a **dyad** or **tensor product** or **open product** of vectors $\mathbf{a}$ and $\mathbf{b}$. A dyad $\mathbf{ab}$ is defined as that associates each vector $\mathbf{v}$ with a vector $(\mathbf{b}\cdot\mathbf{v})\mathbf{a}$, that is, $(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a}$, where the point denotes a scalar product.

Following an analogous interpretation for $\mathbf{j}\cdot\phi$ i $\mathbf{k}\cdot\phi$, we finally obtain

$$\begin{aligned}\mathbf{A}\cdot\phi =& A_1(a_{11}\mathbf{i}+a_{12}\mathbf{j}+a_{13}\mathbf{k})+A_2(a_{21}\mathbf{i}+a_{22}\mathbf{j}+a_{23}\mathbf{k})+\\ &+A_3(a_{31}\mathbf{i}+a_{32}\mathbf{j}+a_{33}\mathbf{k})=\\ =&(A_1a_{11}+A_2a_{21}+A_3a_{31})\mathbf{i}+(A_1a_{12}+A_2a_{22}+A_3a_{32})\mathbf{j}+\\ &+(A_1a_{13}+A_2a_{23}+A_3a_{33})\mathbf{k}.\end{aligned}$$

Thus, the scalar product of vector and dyad $\mathbf{A}\cdot\phi$ is a vector.

Exercise 74

Let the vector functions $\mathbf{A}$ and $\mathbf{B}$, the scalar function ϕ, and the operator ∇, be expressed with respect to the Cartesian coordinate system.

a) Interpret the symbol $\mathbf{A}\cdot\nabla$, and then apply it to the scalar function ϕ.
b) Give a possible definition of $(\mathbf{A}\cdot\nabla)\mathbf{B}$.
c) Is it possible to write this expression without brackets, as $\mathbf{A}\cdot\nabla\mathbf{B}$, without changing its sense?

Solution

a) Let $\mathbf{A}$ be expressed as $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$. Then

$$\begin{aligned}\mathbf{A}\cdot\nabla =&(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\\ =&A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\end{aligned}$$

represents an operator. By applying this operator to the function ϕ, we obtain

$$(\mathbf{A}\cdot\nabla)\phi=\left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\phi=A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}$$

This is the same as $\mathbf{A}\cdot(\nabla\phi)$. Thus, given that

$$(\mathbf{A}\cdot\nabla)\phi=\mathbf{A}\cdot(\nabla\phi),$$

in this case the brackets can be omitted, that is

$$(\mathbf{A}\cdot\nabla)\phi=\mathbf{A}\cdot(\nabla\phi)=\mathbf{A}\cdot\nabla\phi.$$

b) If we replace ϕ by $\mathbf{B}=B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}$ in a) we obtain

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\mathbf{B}=&\left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\mathbf{B}=\\ =&A_1\frac{\partial\mathbf{B}}{\partial x}+A_2\frac{\partial\mathbf{B}}{\partial y}+A_3\frac{\partial\mathbf{B}}{\partial z}=\\ =&\left(A_1\frac{\partial B_1}{\partial x}+A_2\frac{\partial B_1}{\partial y}+A_3\frac{\partial B_1}{\partial z}\right)\mathbf{i}+\\ +&\left(A_1\frac{\partial B_2}{\partial x}+A_2\frac{\partial B_2}{\partial y}+A_3\frac{\partial B_2}{\partial z}\right)\mathbf{j}+\\ +&\left(A_1\frac{\partial B_3}{\partial x}+A_2\frac{\partial B_3}{\partial y}+A_3\frac{\partial B_3}{\partial z}\right)\mathbf{k}.\end{aligned}$$

c) Let us now use the previously computed expression for $\nabla\mathbf{B}$ (see Example 72)

$$\begin{aligned}\mathbf{A}\cdot\nabla\mathbf{B} =&(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})\cdot\nabla\mathbf{B} =\\ =&A_1\mathbf{i}\cdot\nabla\mathbf{B}+A_2\mathbf{j}\cdot\nabla\mathbf{B}+A_3\mathbf{k}\cdot\nabla\mathbf{B}\\ =&A_1\left(\frac{\partial B_1}{\partial x}\mathbf{i}+\frac{\partial B_2}{\partial x}\mathbf{j}+\frac{\partial B_3}{\partial x}\mathbf{k}\right)+A_2\left(\frac{\partial B_1}{\partial y}\mathbf{i}+\frac{\partial B_2}{\partial y}\mathbf{j}+\frac{\partial B_3}{\partial y}\mathbf{k}\right)+\\ &+A_3\left(\frac{\partial B_1}{\partial z}\mathbf{i}+\frac{\partial B_2}{\partial z}\mathbf{j}+\frac{\partial B_3}{\partial z}\mathbf{k}\right).\end{aligned}$$

It is obvious from here that the result is the same as the one obtained under b), namely

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \mathbf{A}\cdot(\nabla\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B}.$$

We have used here the results from Example 73 as well.

Exercise 75

If

$$\begin{aligned}\mathbf{A} &= 2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\\ \mathbf{B} &= x^2\mathbf{i}+yz\mathbf{j}-xy\mathbf{k}\\ \phi &= 2x^2yz^3\end{aligned}$$

find

a) $(\mathbf{A}\cdot\nabla)\phi$,
b) $\mathbf{A}\cdot\nabla\phi$,
c) $(\mathbf{B}\cdot\nabla)\mathbf{A}$,
d) $(\mathbf{A}\times\nabla)\phi$,
e) $\mathbf{A}\times\nabla\phi$.

Solution

a)

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\,\phi &= \left[\left(2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\right)\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\right]\phi =\\ &=\left(2yz\frac{\partial}{\partial x}-x^2y\frac{\partial}{\partial y}+xz^2\frac{\partial}{\partial z}\right)\left(2x^2yz^3\right) =\\ &=2yz\frac{\partial}{\partial x}\left(2x^2yz^3\right)-x^2y\frac{\partial}{\partial y}\left(2x^2yz^3\right)+xz^2\frac{\partial}{\partial z}\left(2x^2yz^3\right) =\\ &=(2yz)\left(4xyz^3\right)-\left(x^2y\right)\left(2x^2z^3\right)+\left(xz^2\right)\left(6x^2yz^2\right) =\\ &=8xy^2z^4-2x^4yz^3+6x^3yz^4.\end{aligned}$$

b)

$$
\begin{aligned}
\mathbf{A}\cdot\nabla\phi &= \left(2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)=\\
&=\left(2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\right)\cdot\left(4xyz^3\mathbf{i}+2x^2z^3\mathbf{j}+6x^2yz^2\mathbf{k}\right)=\\
&=8xy^2z^4-2x^4yz^3+6x^3yz^4
\end{aligned}
$$

The result is the same as the result under a). Thus, $(\mathbf{A}\cdot\nabla)\,\phi=\mathbf{A}\cdot\nabla\phi$, the same as in Example 74, on p.145.

c)

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\,\mathbf{A} &= \left[\left(x^2\mathbf{i}+yz\mathbf{j}-xy\mathbf{k}\right)\cdot\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\right]\mathbf{A}=\\
&=\left(x^2\frac{\partial}{\partial x}+yz\frac{\partial}{\partial y}-xy\frac{\partial}{\partial z}\right)\mathbf{A}=\\
&=\left(x^2\frac{\partial\mathbf{A}}{\partial x}+yz\frac{\partial\mathbf{A}}{\partial y}-xy\frac{\partial\mathbf{A}}{\partial z}\right)=\\
&=x^2\left(-2xy\mathbf{j}+z^2\mathbf{k}\right)+yz\left(2z\mathbf{i}-x^2yz\mathbf{j}\right)-xy\left(2y\mathbf{i}+2xz\mathbf{k}\right)=\\
&=\left(2yz^2-2xy^2\right)\mathbf{i}-\left(2x^3y+x^2yz\right)\mathbf{j}+\left(x^2z^2-2x^2yz\right)\mathbf{k}.
\end{aligned}
$$

d)

$$
\begin{aligned}
(\mathbf{A}\times\nabla)\,\phi &= \left[\left(2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\right)\times\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\right]\phi=\\
&=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2yz & -x^2y & xz^2\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\end{vmatrix}\phi=\\
&=\left[\left(-x^2y\frac{\partial}{\partial z}-xz^2\frac{\partial}{\partial y}\right)\mathbf{i}+\left(xz^2\frac{\partial}{\partial x}-2yz\frac{\partial}{\partial z}\right)\mathbf{j}+\right.\\
&\qquad\left.+\left(2yz\frac{\partial}{\partial y}+x^2y\frac{\partial}{\partial x}\right)\mathbf{k}\right]\phi=\\
&=\left(-x^2y\frac{\partial\phi}{\partial z}-xz^2\frac{\partial\phi}{\partial y}\right)\mathbf{i}+\left(xz^2\frac{\partial\phi}{\partial x}-2yz\frac{\partial\phi}{\partial z}\right)\mathbf{j}+\\
&\qquad+\left(2yz\frac{\partial\phi}{\partial y}+x^2y\frac{\partial\phi}{\partial x}\right)\mathbf{k}=\\
&=-\left(6x^4y^2z^2+2x^3z^5\right)\mathbf{i}+\left(4x^2yz^5-12x^2y^2z^3\right)\mathbf{j}+\\
&\qquad+\left(4x^2yz^4+4x^3y^2z^3\right)\mathbf{k}.
\end{aligned}
$$

e)

$$\begin{aligned}
\mathbf{A}\times\nabla\phi &= \left(2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2yz & -x^2y & xz^2 \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix} = \\
&= \left(-x^2y\frac{\partial\phi}{\partial z} - xz^2\frac{\partial\phi}{\partial y}\right)\mathbf{i} + \left(xz^2\frac{\partial\phi}{\partial x} - 2yz\frac{\partial\phi}{\partial z}\right)\mathbf{j} + \\
&\qquad + \left(2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x}\right)\mathbf{k} = \\
&= -\left(6x^4y^2z^2 + 2x^3z^5\right)\mathbf{i} + \left(4x^2yz^5 - 12x^2y^2z^3\right)\mathbf{j} + \\
&\qquad + \left(4x^2yz^4 + 4x^3y^2z^3\right)\mathbf{k}
\end{aligned}$$

Comparing with d) we can see that $(\mathbf{A}\times\nabla)\,\phi = \mathbf{A}\times\nabla\phi$.

4.6.5 Invariant

Exercise 76

Two cartesian coordinate systems xyz and $x'y'z'$, with a common coordinate origin, rotate relative to each other. Determine the connections between the coordinates of an arbitrary point P expressed in relation to these two coordinate systems (coordinate transformations).

Solution

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of point P in these two systems. Given that these coordinate systems have the same origin, it follows that $\mathbf{OA} = \mathbf{r} = \mathbf{r}'$, and expressed by their components (in relation to the coordinate systems)

$$x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{4.178}$$

If we multiply equation (4.178) alternately with $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, we obtain

$$\begin{aligned} x' &= \ell_{11}x + \ell_{12}y + \ell_{13}z, \\ y' &= \ell_{21}x + \ell_{22}y + \ell_{23}z, \\ z' &= \ell_{31}x + \ell_{32}y + \ell_{33}z, \end{aligned} \tag{4.179}$$

where $\ell_{11} = \mathbf{i}' \cdot \mathbf{i}$, $\ell_{12} = \mathbf{i}' \cdot \mathbf{j}$, $\dots$, $\ell_{33} = \mathbf{k}' \cdot \mathbf{k}$.

Exercise 77

Prove that

$$\begin{aligned} \mathbf{i}' &= \ell_{11}\mathbf{i} + \ell_{12}\mathbf{j} + \ell_{13}\mathbf{k}, \\ \mathbf{j}' &= \ell_{21}\mathbf{i} + \ell_{22}\mathbf{j} + \ell_{23}\mathbf{k}, \\ \mathbf{k}' &= \ell_{31}\mathbf{i} + \ell_{32}\mathbf{j} + \ell_{33}\mathbf{k}. \end{aligned}$$

Solution

Each vector $\mathbf{v}'$ in the system S' can be expressed by means of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ of the system S as follows

$$\mathbf{v}' = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k},$$

which is also valid for the unit vectors of the system S', namely

$$\mathbf{i}' = \alpha_1\mathbf{i} + \beta_1\mathbf{j} + \gamma_1\mathbf{k}, \tag{4.180}$$
$$\mathbf{j}' = \alpha_2\mathbf{i} + \beta_2\mathbf{j} + \gamma_2\mathbf{k},$$
$$\mathbf{k}' = \alpha_3\mathbf{i} + \beta_3\mathbf{j} + \gamma_3\mathbf{k}.$$

Let us first determine the coefficients $\alpha_1, \beta_1, \gamma_1$. By multiplying equation (4.180) alternately with **i**, **j**, **k**, we obtain

$$\mathbf{i}' \cdot \mathbf{i} = \alpha_1 = \ell_{11},$$
$$\mathbf{i}' \cdot \mathbf{j} = \beta_1 = \ell_{12},$$
$$\mathbf{i}' \cdot \mathbf{k} = \gamma_1 = \ell_{13}.$$

We have thus proved the first required equation

$$\mathbf{i}' = \ell_{11}\mathbf{i} + \ell_{12}\mathbf{j} + \ell_{13}\mathbf{k}.$$

The remaining two equations can be proved analogously.

Note that for a clearer representation of these relations, the dependencies can be tabulated.

	$\mathbf{i}_1$	$\mathbf{j}_2$	$\mathbf{k}_3$
$\mathbf{i}'_1$	ℓ_{11}	ℓ_{12}	ℓ_{13}
$\mathbf{j}'_2$	ℓ_{21}	ℓ_{22}	ℓ_{23}
$\mathbf{k}'_3$	ℓ_{31}	ℓ_{32}	ℓ_{33}

Exercise 78

Prove that

$$\sum_{p=1}^{3} \ell_{pm}\ell_{pn} = \begin{cases} 1, & \text{for} \quad m = n, \\ 0, & \text{for} \quad m \neq n, \end{cases}$$

m and n take the values $1,2,3$.

Solution

Similarly as in Example 77, it can be proved that

$$\mathbf{i} = \ell_{11}\mathbf{i}' + \ell_{21}\mathbf{j}' + \ell_{31}\mathbf{k}',$$
$$\mathbf{j} = \ell_{12}\mathbf{i}' + \ell_{22}\mathbf{j}' + \ell_{32}\mathbf{k}',$$
$$\mathbf{k} = \ell_{13}\mathbf{i}' + \ell_{23}\mathbf{j}' + \ell_{33}\mathbf{k}',$$

and it thus follows that

$$\mathbf{i}\cdot\mathbf{i} = 1 = (\ell_{11}\mathbf{i}' + \ell_{21}\mathbf{j}' + \ell_{31}\mathbf{k}')\cdot(\ell_{11}\mathbf{i}' + \ell_{21}\mathbf{j}' + \ell_{31}\mathbf{k}') =$$
$$= \ell_{11}^2 + \ell_{21}^2 + \ell_{31}^2 = \sum_{p=1}^{3} \ell_{p1}\ell_{p1},$$
$$\mathbf{i}\cdot\mathbf{j} = 0 = (\ell_{11}\mathbf{i}' + \ell_{21}\mathbf{j}' + \ell_{31}\mathbf{k}')\cdot(\ell_{12}\mathbf{i}' + \ell_{22}\mathbf{j}' + \ell_{32}\mathbf{k}') =$$
$$= \ell_{11}\ell_{12} + \ell_{21}\ell_{22} + \ell_{13}\ell_{32} = \sum_{p=1}^{3} \ell_{p1}\ell_{p2},$$
$$\mathbf{i}\cdot\mathbf{k} = 0 = (\ell_{11}\mathbf{i}' + \ell_{21}\mathbf{j}' + \ell_{31}\mathbf{k}')\cdot(\ell_{13}\mathbf{i}' + \ell_{23}\mathbf{j}' + \ell_{33}\mathbf{k}') =$$
$$= \ell_{11}\ell_{13} + \ell_{21}\ell_{23} + \ell_{31}\ell_{33} = \sum_{p=1}^{3} \ell_{p1}\ell_{p3}.$$

Thus, we have proved that the initial relation is valid for $m = 1$. The validity of the relation for $m = 2,3$ and $n = 1,2,3$ can be proved analogously.

Using the Kronecker symbol

$$\delta_{mn} = \begin{cases} 1, & \text{za} \quad m = n, \\ 0, & \text{za} \quad m \neq n, \end{cases}$$

we can rewrite the previous result in the following form

$$\sum_{p=1}^{3} \ell_{pm}\ell_{pn} = \delta_{mn}.$$

Exercise 79

If $\mathbf{v} = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$ i $\phi = 2z - x^3y$, find

a) $\mathbf{v} \cdot \nabla\phi$,

b) $\mathbf{v} \times \nabla\phi$

in point $A(1,-1,1)$.

Solution

a) 5, b) $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$.

Exercise 80

Prove that $\nabla f(r) = \dfrac{f'(r)\mathbf{r}}{r}$.

Exercise 81

If U is a differentiable function of variables x, y, and z prove that

$$\nabla U \cdot \mathrm{d}\mathbf{r} = \mathrm{d}U.$$

Exercise 82

Let F be a differentiable scalar function of variables x, y, z, and t, where x, y, and z are differentiable functions of t. Prove that

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Exercise 83

If $\mathbf{v}$ is a constant vector, prove that $\nabla(\mathbf{r}\cdot\mathbf{v}) = \mathbf{v}$.

Exercise 84

If $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, prove that

$$\mathrm{d}\mathbf{v} = (\nabla v_1 \cdot \mathrm{d}\mathbf{r})\mathbf{i} + (\nabla v_2 \cdot \mathrm{d}\mathbf{r})\mathbf{j} + (\nabla v_3 \cdot \mathrm{d}\mathbf{r})\mathbf{k}.$$

Exercise 85

Find the increment of the function $\phi = 4xz^3 - 3x^2y^2z$ in point $A(2,-1,2)$ in the direction of vector $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

Solution

$376/7$.

Exercise 86

Find the increment of the function $\varphi = 4e^{2x-y+z}$ in point $A(1,1,-1)$ in the direction of point $B(-3,5,6)$.

Solution

$-20/9$.

Exercise 87

In the direction of which vector, from point $A(1,3,2)$, is the increment of the function $\phi = 2xz - y^2$ the largest? What is that increment?

Solution

$\mathbf{v} = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}, \qquad 2\sqrt{14}$.

Exercise 88

Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ in point $A(1,-2,1)$.

Solution

$\arccos \dfrac{\sqrt{6}}{14}$.

Exercise 89

Find the constants a and b such that the surface $ax^2 - byz = (a+2)x$ is normal to the surface $4x^2y + z^3 = 4$ at point $M(1,-1,2)$.

Solution

$a = 5/2$, $b = 1$.

Exercise 90

a) Prove that the functions $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ are functionally dependent, that is, $(F(u,v,w) = 0)$ iff $\nabla u \cdot \nabla v \times \nabla w = 0$.
b) Express $\nabla u \cdot \nabla v \times \nabla w = 0$ in the form of a determinant.
c) Are the functions $u = x+y+z$, $v = x^2+y^2+z^2$ and $w = xy+yz+zx$ dependent?

Solution

b) $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$,

c) Yes, as in this case $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+z \end{vmatrix} = 0$. Their functional dependence is of the from $u^2 - v - 2w = 0$.

Exercise 91

If $\mathbf{A} = 3xyz^2\mathbf{i} + 2xy^3\mathbf{j} - x^2yz\mathbf{k}$ i $\phi = 3x^2 - yz$ find

a) $\nabla \cdot \mathbf{A}$,
b) $\mathbf{A} \cdot (\nabla\phi)$,
c) $\nabla \cdot (\phi\mathbf{A})$,
d) $\nabla \cdot (\nabla\phi)$, at point (1,-1,1).

Solution

a) 4, b) -15, c) 1, d) 6.

Exercise 92

a) Prove that $\nabla^2 f(r) = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}$.
b) Find $f(r)$ so that $\nabla^2 f(r) = 0$.

Solution

b) $f(r) = A + B/r$, where A and B are constants.

Exercise 93

Prove that the vector field

$$\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$$

is not a solenoid field, whereas $\mathbf{B} = xyz^2\mathbf{A}$ is.

4.6.6 Integrals, integral theorems

Exercise 94

Find the area of the ellipse.

Solution

The area P of any flat surface S is equal to

$$P = \iint_S \mathrm{d}x\mathrm{d}y.$$

Let the surface S be bounded by a closed curve C, then according to Stokes' theorem[21]. By adding these two integrals we obtain

$$P = \iint_S \mathrm{d}x\mathrm{d}y = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

By switching to polar coordinates $x = a\cos\theta$, $y = b\sin\theta$, we obtain

$$P = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta =$$

$$\frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi} abd\theta = \pi ab.$$

Exercise 95

Let us observe the motion of a material point (particle) M under the action of a central force[22]. The differential equation of motion of a particle M, of mass m, can then be represented in the form

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = f(r)\,\mathbf{r}_0,$$

where $\mathbf{r}$ is the position vector of the particle M measured in relation to the coordinate origin O, $\mathbf{r}_0$ is the unit vector of vector $\mathbf{r}$, and $f(r)$ is a function of the distance of point M from O.

a) Give an interpretation of the physical meaning of the expressions $f(r) < 0$ and $f(r) > 0$.
b) Prove that $\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{c}$, where $\mathbf{c}$ is a constant vector.
c) Give a geometric an kinematic interpretation of the result under b).
d) Relate the previous result to the motion of the planets in our solar system.

Solution

a) For $f(r) < 0$, the acceleration $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ is a vector opposite to vector $\mathbf{r}$, and thus the force has direction from M to O, which means that an attractive force acts on the particle from point O.
If $f(r) > 0$, then the force has direction from O to M, and a repulsive force acts on the particle from point O.
b) Let us form a vector product of both sides of expression $m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = f(r)\,\mathbf{r}_0$ with vector $\mathbf{r}$, from the left. We thus obtain

$$m\mathbf{r} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = f(r)\,\mathbf{r} \times \mathbf{r}_0 = \mathbf{0},$$

as the vectors $\mathbf{r}$ and $\mathbf{r}_0$ are collinear, that is, $\mathbf{r} \times \mathbf{r}_0 = \mathbf{0}$. From here it follows that

$$m\mathbf{r} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{0},$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) = \mathbf{0}.$$

By integration of both sides we obtain

$$\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{c}, \tag{4.181}$$

where $\mathbf{c}$ is a constant vector.

[21] $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y$. In the special case, when $P = 0$ and $Q = x$, we obtain $\iint_S \mathrm{d}x\,\mathrm{d}y = \oint_C x\,\mathrm{d}y$. In the case when $P = -y$ and $Q = 0$, we obtain $\iint_S \mathrm{d}x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x$.

[22] A force whose direction passes through a fixed point O of space (center of force) is called a central force. A particularly important category of central forces are those whose intensity depends only on the distance from the center.

c) In the time interval Δt the particle moved from position M to position N. The area "swept out" by the position vector, in that time interval, is approximately1 equal to half the area of the parallelogram with sides $\mathbf{r}$ and $\Delta\mathbf{r}$, that is, $\frac{1}{2}\mathbf{r}\times\Delta\mathbf{r}$. The approximate area "swept out" by the position vector in a unit of time is thus equal to $\frac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}$. Hence the instantaneous change in surface area over time is equal to

$$\lim_{\Delta t\to 0}\left(\frac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}\right)=\frac{1}{2}\mathbf{r}\times\mathbf{v},$$

where $\mathbf{v}$ is the current velocity of the particle. The value $\mathbf{H}=\frac{1}{2}\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\frac{1}{2}\mathbf{r}\times\mathbf{v}$ is called sector velocity. As the sector velocity is constant, in the case of central forces (see (4.181)), we obtain

$$\mathbf{H}=\frac{1}{2}\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathbf{c},$$

where $\mathbf{c}$ is a vector constant. Given that $\mathbf{r}\cdot\mathbf{H}=\mathbf{r}\cdot\left(\frac{1}{2}\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)=0$ (condition of coplanarity) it follows that the point moves in a plane. Namely, given that $\mathbf{H}$ is constant, it follows that this vector does not change its direction. However, the vector is perpendicular to the plane formed by the vectors $\mathbf{r}$ and $\mathbf{v}$, so it can be concluded that the point moves in a plane with its normal in the direction of $\mathbf{H}$.

d) A planet (such as the Earth) is attracted by the Sun according to Newton's law of gravity which reads: *any two bodies of mass m and M are attracted to each other by a force equal to*

$$\mathbf{F}=\frac{\gamma Mm}{r^2}\mathbf{r}_0,$$

where r is the distance between the centers of the two bodies, and γ the universal gravitational constant. Let m and M be the masses of the planet and the Sun, respectively. Let us chose a coordinate system with its origin in O, placed in the center of the Sun. Then the equation of motion of the planet under the influence of the Sun is

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}=-\frac{\gamma mM}{r^2}\mathbf{r}_0,$$

or

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}=-\frac{\gamma M}{r^2}\mathbf{r}_0.$$

According to c) the planet moves around the Sun so that its position vector sweeps out equal areas during equal intervals of time. This law is known in the literature as Kepler's second law (law of equal areas).

Exercise 96

Show that a planet moves around the Sun in an ellipse, where the Sun is in the focus of that ellipse.

Solution

Given that

$$\mathbf{r} = r\mathbf{r}_0,$$

it follows that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} = r\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\mathbf{r}_0.$$

From the previous Example, we have

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\gamma M}{r^2}\mathbf{r}_0, \tag{4.182}$$

as well as

$$\mathbf{r} \times \mathbf{v} = 2\mathbf{H} = \mathbf{h}. \tag{4.183}$$

It follows that

$$\mathbf{h} = r\mathbf{r}_0 \times \left(r\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\mathbf{r}_0 \right) = r^2\,\mathbf{r}_0 \times \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t}. \tag{4.184}$$

Let us form a vector product of both sides of the equation (4.182) with vector $\mathbf{h}$, from the right. Using (4.184) and Example 5d on p. 58, we then obtain

$$\begin{aligned} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \times \mathbf{h} &= -\frac{\gamma M}{r^2}\mathbf{r}_0 \times \mathbf{h} = -\gamma M\,\mathbf{r}_0 \times \left(\mathbf{r}_0 \times \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} \right) \\ &= -\gamma M \left[\left(\mathbf{r}_0 \cdot \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} \right) \mathbf{r}_0 - (\mathbf{r}_0 \cdot \mathbf{r}_0)\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} \right] = \gamma M \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t}, \end{aligned} \tag{4.185}$$

because $\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t} \cdot \mathbf{r}_0 = 0$ (see Example 26 on p. 69). Given that $\mathbf{h}$ is a constant vector, it follows that

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \times \mathbf{h} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \times \mathbf{h}),$$

and then, based on (4.185),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \times \mathbf{h}) = \gamma M \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}t}. \tag{4.186}$$

Integration of the relation (4.186) yields

$$\mathbf{v} \times \mathbf{h} = \gamma M \mathbf{r}_0 + \mathbf{p},$$

where $\mathbf{p}$ is an arbitrary constant vector. From here, by a scalar product with $\mathbf{r}$, we obtan

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) =& \gamma M \mathbf{r} \cdot \mathbf{r}_0 + \mathbf{r} \cdot \mathbf{p} = \\ =& \gamma M r + r\mathbf{r}_0 \cdot \mathbf{p} = \gamma M r + rp\cos\theta \end{aligned}$$

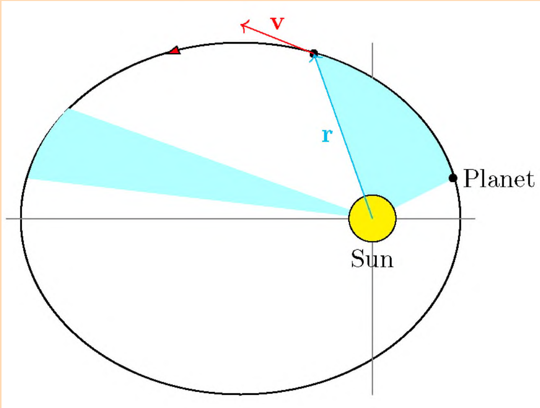


Figure 4.27: Movement of a planet around the Sun.

where θ is the angle between $\mathbf{p}$ and $\mathbf{r}_0$, and p is the magnitude of vector $\mathbf{p}$.

Given that

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) = \mathbf{r}\times\mathbf{v}\cdot\mathbf{h} = \mathbf{h}\cdot\mathbf{h} = h^2,$$

we have $h^2 = \gamma M r + r p\cos\theta$ and

$$r = \frac{h^2}{\gamma M + p\cos\theta} = \frac{h^2/\gamma M}{1+(p/\gamma M)\cos\theta}.$$

This equation represents the equation of the ellipse with respect to the polar cylindrical coordinate system, with its origin at the center of the ellipse.

Exercise 97

If $\mathbf{v} = (3x^2+6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, determine $\int_c \mathbf{v}\cdot \mathrm{d}\mathbf{r}$ from point $O(0,0,0)$ to point $A(1,1,1)$ along the curve c determined by

a) parametric equations $x=t$, $y=t^2$, $z=t^3$,

b) the line $OBCA$, composed of segments OB, BC and CA, where the coordinates of the points on the line are as follows: $O(0,0,0)$, $B(1,0,0)$, $C(1,1,0)$, $A(1,1,1)$,

c) a straight line passing through points $O(0,0,0)$ and $A(1,1,1)$.

Solution

For the given vector field $\mathbf{v}$ the following stands

$$\int_c \mathbf{v}\cdot\mathrm{d}\mathbf{r} = \int_c [(3x^2+6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}]\cdot(\mathrm{d}x\,\mathbf{i}+\mathrm{d}y\,\mathbf{j}+\mathrm{d}z\,\mathbf{k})$$

$$= \int_c (3x^2+6y)\,\mathrm{d}x - 14yz\,\mathrm{d}y + 20xz^2\,\mathrm{d}z.$$

a) Parameter values $t=0$ and $t=1$ correspond to the points $O(0,0,0)$ and $A(1,1,1)$

on the curve c, respectively. It follows that

$$\begin{aligned}\int_c \mathbf{v}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (3t^2+6t^2)\,\mathrm{d}t - 14(t^2)(t^3)\,\mathrm{d}(t^2) + 20(t)(t^3)^2\,\mathrm{d}(t^3)\\ &= \int_0^1 9t^2\mathrm{d}t - 28t^6\,\mathrm{d}t + 60t^9\,\mathrm{d}t\\ &= \int_0^1 (9t^2 - 28t^6t + 60t^9)\,\mathrm{d}t\\ &= 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5.\end{aligned}$$

b) Given that

$$\int_c = \int_{c_1} + \int_{c_2} + \int_{c_3}$$

where $c = c_1 \cup c_2 \cup c_3$, we will divide the line $OBCA$ as follows
- segment c_1, connecting points $O(0,0,0)$ and $B(1,0,0)$, and lying on the x axis, which means that $y=0$, $z=0$, $\mathrm{d}y=0$, $\mathrm{d}z=0$, while x takes the values from 0 to 1;
- segment c_2, connecting points $B(1,0,0)$ and $C(1,1,0)$, and lying in the xy plane, parallel to the y axis, which means that $x=1$, $z=0$, $\mathrm{d}x=0$, $\mathrm{d}z=0$, while y takes the values from 0 to 1;
- segment c_3, connecting points $C(1,1,0)$ i $A(1,1,1)$, and lying in a plane parallel to the plane yz and parallel to the z axis, which means that $x=1$, $y=1$, $\mathrm{d}x=0$, $\mathrm{d}y=0$, while z takes the values from 0 to 1.

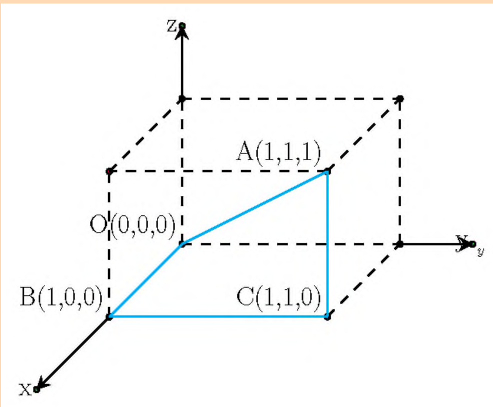


Figure 4.28

Let us now calculate the integrals along this segments, namely c_1, c_2 and c_3.

$$I_1 = \int_{c_1} = \int_0^1 (3x^2 + 6(0))\,\mathrm{d}x - 14(0)(0)\,0 + 20x(0)^2\,0 = \int_0^1 3x^2\,\mathrm{d}x = x^3\Big|_0^1 = 1,$$

$$I_2 = \int_{c_2} = \int_0^1 (3(1)^2 + 6(y))\,0 - 14y(0)\,\mathrm{d}y + 20(1)(0)^2\,0 = 0,$$

$$I_3 = \int_{c_3} = \int_0^1 (3(1)^2 + 6(y))\,0 - 14(1)(0)\,0 + 20(1)(z)^2\,\mathrm{d}z = \int_0^1 20z^2\,\mathrm{d}z = \frac{20z^3}{3}\Big|_0^1 = \frac{20}{3}.$$

Finally, adding the three integrals we obtain

$$I = I_1 + I_2 + I_3 = \int_c \mathbf{v}\cdot\mathbf{dr} = 1 + 0 + \frac{20}{3} = \frac{23}{3}.$$

c) The straight line, passing through points $O(0,0,0)$ and $A(1,1,1)$, can be represented in parametric form as $x = y = z = t$. Parameter values $t = 0$ and $t = 1$ correspond to points O and A, respectively, and thus it follows

$$\int_c \mathbf{v}\cdot\mathbf{dr} = \int_0^1 (3t^2 + 6t)\,\mathrm{d}t - 14(t)(t)\,\mathrm{d}t + 20(t)(t)^2\,\mathrm{d}t$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)\,\mathrm{d}t = \int_0^1 (6t - 11t^2 + 20t^3)\,\mathrm{d}t = \frac{13}{3}.$$

Note that these examples show that the values of the curvilinear integrals depend on the path (line) along which the integrals are calculated, and which passes through the given points.

Exercise 98

Observe the force $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$. Calculate the work of this force $\int_c \mathbf{F}\cdot\mathbf{dr}$ along the curve c in the xy plane, given by the equation $y = 2x^2$, from point $O(0,0)$ to point $A(1,2)$.

Solution

Given that the integration is performed in the xy plane, it follows that $\mathbf{dr} = \mathrm{d}x\,\mathbf{i} + \mathrm{d}y\,\mathbf{j}$, and the force is

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_c (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) =$$

$$= \int_c 3xy\,dx - y^2\,dy.$$

Let us introduce the parameter t such that $x = t$. The parametric equation of the curve c is $x = t$ and $y = 2t^2$. Parameter values $t = 0$ and $t = 1$ correspond to points $O(0,0)$ and $A(1,2)$, respectively. Then $dx = dt$ and $dy = 4t\,dt$, and it follows that

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_0^1 3(t)(2t^2)\,dt - (2t^2)^2\,4t\,dt =$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt = -\frac{7}{6}.$$

Exercise 99

Let $\mathbf{F} = \nabla\phi$, where ϕ is an unambiguous function with continuous second order partial derivatives.

a) Show that the integral $\int_c \mathbf{F}\cdot d\mathbf{r}$, calculated between points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, does not depend on the choice of the path between these two points.

b) Conversely, if the integral $\int_c \mathbf{F}\cdot d\mathbf{r}$ is independent of the path c between any two points, then there exists a function ϕ such that $\mathbf{F} = \nabla\phi$. Prove this.

Solution

a) Let us denote the integral $\int_c \mathbf{F}\cdot d\mathbf{r}$ between points P_1 and P_2 by A_{12}, that is

$$A_{12} = \int_c \mathbf{F}\cdot d\mathbf{r}.$$

According to the initial assumption $\mathbf{F} = \nabla\phi$, and it follows that

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_c \nabla\phi\cdot d\mathbf{r} =$$

$$= \int_c d\phi,$$

where $d\phi$ is a total differential (see Example 81, p.151), and if further follows that

$$A_{12} = \phi(P_2) - \phi(P_1) = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1).$$

Thus, the integral A_{12} depends on the start and end point of the path, but not on the path between these points itself.

b) Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then the integral along the curve c, between points $P_1(x_1, y_1, z_1)$ and $P(x,y,z)$, is

$$\phi(x,y,z) = \int\limits_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int\limits_{(x_1,y_1,z_1)}^{(x,y,z)} F_1 dx + F_2 dy + F_3 dz.$$

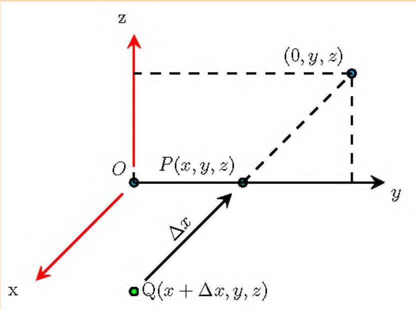
From here it follows

$$\phi(x+\Delta x,y,z) - \phi(x,y,z) = \int\limits_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} - \int\limits_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} =$$

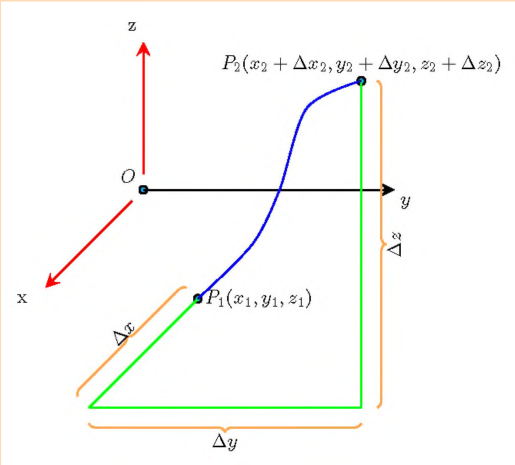
$$= \int\limits_{(x,y,z)}^{(x_1,y_1,z_1)} \mathbf{F}\cdot d\mathbf{r} + \int\limits_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} =$$

$$= \int\limits_{(x,y,z)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int\limits_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 dx + F_2 dy + F_3 dz.$$

Since the last integral, according to the assumption, does not depend on the path between the points with coordinates (x,y,z) and $(x+\Delta x, y, z)$, we can choose a straight line that passes through these two points as the path, so that $dy = dz = 0$ (the line is parallel to the x axis, see Figure 4.29b). Then



(a) Work along the x axis.



(b) Work from point P_1 to point P_2.

Figure 4.29

$$\phi(x+\Delta x,y,z) - \phi(x,y,z) = \int\limits_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 dx.$$

From here we obtain

$$\frac{\phi(x+\Delta x,y,z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x} \int\limits_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 dx. \qquad (4.187)$$

If we now apply the mean value theorem to the previous integral, we obtain

$$\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 \mathrm{d}x = \Delta x F_1(x+\theta\Delta x, y, z), \quad 0 < \theta < 1. \tag{4.188}$$

By substituting the relation (4.188) into (4.187), and letting $\Delta x \to 0$, we obtain

$$\frac{\partial \phi}{\partial x} = F_1. \tag{4.189}$$

Analogously, we obtain $\dfrac{\partial \phi}{\partial y} = F_2$ and $\dfrac{\partial \phi}{\partial z} = F_3$. Finally

$$\mathbf{F} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \nabla \phi.$$

Note. If $\mathbf{F}$ is a force field, then in mechanics the integral $\int_c \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ represents the work of the force. Forces whose work does not depend on the trajectories, which pass through points P_1 and P_2, are called **conservative forces**, and the corresponding field is called a **conservative field**.

Exercise 100

Observe the vector field $\mathbf{F}$.

a) Let the integral $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ be independent of the path. Prove that in that case

$$\operatorname{rot} \mathbf{F} (= \nabla \times \mathbf{F}) = \mathbf{0}.$$

b) Conversely, if $\nabla \times \mathbf{F} = \mathbf{0}$, prove that $\mathbf{F}$ is a conservative field.

Proof

a) If $\mathbf{F}$ is a conservative field, then $\mathbf{F} = \nabla \phi$, according to (4.56) (see p. 93), and it follows that $\operatorname{rot} \mathbf{F} = \mathbf{0}$ (see (4.69), p. 98).

b) Conversely, if $\nabla \times \mathbf{F} = \mathbf{0}$, then

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{0},$$

that is

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \tag{*}$$

It follows from the first of these two equations that

$$F_3 = \frac{\partial \varphi}{\partial z} \quad \text{i} \quad F_2 = \frac{\partial \varphi}{\partial y}, \quad \text{where } \varphi = \varphi(x, y, z).$$

We can write the second equation in the form

$$\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \varphi}{\partial x}$$

from where it follows that

$$F_1 = \frac{\partial \varphi}{\partial x}.$$

It further follows that

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \nabla\phi.$$

Thus, the field $\mathbf{F}$ is conservative iff rot$\mathbf{F} = \nabla \times \mathbf{F} = 0$.

Exercise 101

Show that the force $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is conservative and find the scalar potential of this field. Calculate the force necessary to move an object (material point) from point $P(1,-2,1)$ to point $Q(3,1,4)$.

Solution

Given that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \mathbf{0}$$

it follows (see definition (4.47) on p. 92) that the force $\mathbf{F}$ is conservative, that is, that the vector field $\mathbf{F}$ is potential, and thus $\mathbf{F} = \nabla\phi$ or $\nabla\phi = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$.

$$\frac{\partial \phi}{\partial x} = 2xy + z^3 \tag{4.190}$$

$$\frac{\partial \phi}{\partial y} = x^2 \tag{4.191}$$

$$\frac{\partial \phi}{\partial z} = 3xz^2. \tag{4.192}$$

By integrating (4.190) we obtain

$$\phi = x^2y + xz^3 + f(y,z). \tag{4.193}$$

By differentiating the equation (4.193) by y, we obtain

$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial f(y,z)}{\partial y}. \tag{4.194}$$

From equations (4.194) and (4.191) it follows that

$$\frac{\partial f(y,z)}{\partial y} = 0. \tag{4.195}$$

By solving the equation (4.195) we obtain

$$f(y,z) = c_1 + g(z), \tag{4.196}$$

where c_1 is the integration constant. From (4.196) and (4.193) we obtain

$$\phi = x^2 y + xz^3 + g(z) + c_1, \tag{4.197}$$

By differentiating the equation (4.197) by z we obtain

$$\frac{\partial f(y,z)}{\partial z} = 3xz^2 + \frac{\partial g(z)}{\partial z}. \tag{4.198}$$

From equations (4.192) and (4.198) we obtain

$$\frac{\mathrm{d}g(z)}{\mathrm{d}z} = 0. \tag{4.199}$$

By solving the equation (4.199) we obtain

$$g(z) = c_2, \tag{4.200}$$

where c_2 is the integration constant.

By substituting $g(z)$ from (4.200) into equation (4.197) we obtain the final solution

$$\phi = x^2 y + xz^3 + c,$$

where $c = c_1 + c_2$ is an arbitrary constant.

The work A of the force $\mathbf{F}$ is

$$\begin{aligned} A &= \int_P^Q \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_P^Q (2xy + z^3)\mathrm{d}x + x^2 \mathrm{d}y + 3xz^2 \mathrm{d}z \\ &= \int_P^Q \mathrm{d}(x^2 y + xz^3) = x^2 y + xz^3 \Big|_{(1,-2,1)}^{(3,1,4)} = 202 \end{aligned}$$

The work can be expressed as a difference of potentials

$$A = \int_P^Q \nabla\varphi \cdot \mathrm{d}\mathbf{r} = \int_P^Q \mathrm{d}\varphi = \phi(Q) - \phi(P) = 201 - (-1) = 202. \tag{4.201}$$

Exercise 102

Show that if the integral $\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path between any two points P_1 and P_2 of a given region, then $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ for a closed curve, and vice versa.

Solution

Let $P_1AP_2BP_1$ be a closed curve.
Then

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} =$$
$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r} =$$
$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} = 0,$$

because $\int_{P_1AP_2} = \int_{P_1BP_2}$ based on the assumption that the value of the integral does not depend on the path between points P_1 and P_2.

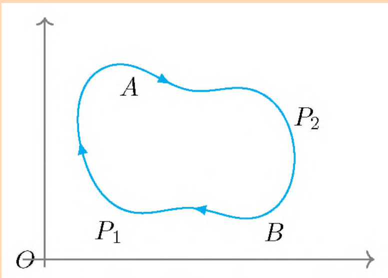


Figure 4.30

Conversely, if $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ then

$$0 = \oint \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r} =$$
$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r}.$$

From there it follows that

$$\int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r}.$$

Exercise 103

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

a) prove that the necessary and sufficient condition for $F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z$ to be a total differential is $\nabla \times \mathbf{F} (= \mathrm{rot}\mathbf{F}) = \mathbf{0}$.

b) Prove that

$$(y^2z^3\cos x - 4x^3z)\mathrm{d}x + 2z^3y\sin x\mathrm{d}y + (3y^2z^2\sin x - x^4)\mathrm{d}z$$

is the total differential of some function ϕ. Find that function.

Solution

a) The condition is necessary. If

$$F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z = \mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z,$$

is the total differential of a function $\phi(x,y,z)$, it follows that

$$F_1 = \frac{\partial\phi}{\partial x}, \quad F_2 = \frac{\partial\phi}{\partial y}, \quad F_3 = \frac{\partial\phi}{\partial z},$$

and thus

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\varphi.$$

According to (see (4.69), p. 98)

$$\nabla \times \mathbf{F} = \nabla \times \nabla\phi = \mathbf{0}.$$

The condition is sufficient. If $\nabla \times \mathbf{F} = 0$, then $\mathbf{F} = \nabla\varphi$. It follows that

$$\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \nabla\varphi\cdot\mathrm{d}\mathbf{r} = \mathrm{d}\varphi$$

(see Example 100 on p.163).

b) Let

$$\mathbf{F} = (y^2z^3\cos x - 4x^3z)\mathbf{i} + 2z^3y\sin x\mathbf{j} + (3y^2z^2\sin x - x^4)\mathbf{k}.$$

Given that, in this case $\nabla \times \mathbf{F} = 0$, then according to the condition under a), there exists a function ϕ such that

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z = F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z,$$

that is

$$(y^2z^3\cos x - 4x^3z)\mathrm{d}x + 2z^3y\sin x\,\mathrm{d}y + (3y^2z^2\sin x - x^4)\mathrm{d}z = \mathrm{d}\phi.$$

It can be proved, analogously to the Example 101 on p.164, that

$$\phi = y^2z^3\sin x - x^4z + \mathrm{const}.$$

Exercise 104

Let $\mathbf{F}$ be a conservative force, that is, $\mathbf{F} = -\nabla\phi$. Suppose a particle of constant mass m moves in its field. If A and B are any two points in that field, prove that

$$\phi(A) + \frac{1}{2}mv_A^2 = \phi(B) + \frac{1}{2}mv_B^2$$

where v_A and v_B are velocities of the particle in points A and B, respectively.

Solution

According to Newton's second law

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}.$$

Given that

$$\mathbf{F}\cdot\mathbf{v} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\mathbf{v} = \frac{m}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v})^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\mathbf{v}^2\right),$$

by integration from point A to point B we obtain

$$\begin{aligned}\int_A^B \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_A^B \mathbf{F}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_A^B \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_A^B \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\mathbf{v}^2\right)\mathrm{d}t = \\ &= \int_A^B \mathrm{d}\left(\frac{1}{2}m\mathbf{v}^2\right) = \frac{m}{2}v^2\Big|_A^B = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2. \end{aligned} \tag{4.202}$$

As $\mathbf{F} = -\nabla\phi$ it follows that

$$\int_A^B \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_A^B \nabla\phi\cdot\mathrm{d}\mathbf{r} = \int_B^A \mathrm{d}\phi = \phi(A) - \phi(B). \tag{4.203}$$

Comparing relations (4.202) and (4.203) we obtain

$$\begin{aligned}&\phi(A) - \phi(B) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \quad \Rightarrow \\ &\phi(A) + \frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2 + \phi(B).\end{aligned}$$

Note that $\phi(A)$ is also called the potential energy at point A, and $\frac{1}{2}mv_A^2$ is the kinetic energy of the particle at point A. The result shows that the total energy, i.e. the sum of the kinetic and potential energy, at point A is equal to the total energy at point B. The law of conservation of energy, in this form, applies only to the fields of conservative forces.

Exercise 105

Let a $\phi = 2xyz^2$ be a scalar function, $\mathbf{F} = xy\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$ a vector field, and c a curve in parametric form $x = t^2$, $y = 2t$, $z = t^3$. Time changes in the interval $t \in [0,1]$. Calculate the following line integrals

a) $\int\limits_c \phi \, d\mathbf{r}$,

b) $\int\limits_c \mathbf{F} \times d\mathbf{r}$.

Solution

a) Let us express ϕ by means of parameter t:

$$\phi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9.$$

Given that

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

it follows that $d\mathbf{r}$, expressed by means of t is

$$d\mathbf{r} = \left(2t\mathbf{i} + 2\mathbf{j} + 3t^2\mathbf{k}\right) dt.$$

It further follows

$$\int\limits_c \phi \, d\mathbf{r} = \int\limits_0^1 4t^9(2t\mathbf{i} + 2\mathbf{j} + 3t^2\mathbf{k})dt =$$

$$= \mathbf{i}\int\limits_0^1 8t^{10}dt + \mathbf{j}\int\limits_0^1 8t^9 dt + \mathbf{k}\int\limits_0^1 12t^{11}dt = \frac{8}{11}\mathbf{i} + \frac{4}{5}\mathbf{j} + \mathbf{k}.$$

b) Analogously to the result under a)

$$\mathbf{F} = xy\mathbf{i} - z\mathbf{j} + x^2\mathbf{k} \quad \Rightarrow \quad \mathbf{F} = 2t^3\mathbf{i} - t^3\mathbf{j} + t^4\mathbf{k}.$$

Then

$$\mathbf{F} \times d\mathbf{r} = (2t^3\mathbf{i} - t^3\mathbf{j} + t^4\mathbf{k}) \times (2t\mathbf{i} + 2\mathbf{j} + 3t^2\mathbf{k})dt =$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = [(-3t^5 - 2t^4)\mathbf{i} + (2t^5 - 6t^5)\mathbf{j} + (4t^3 + 2t^4)\mathbf{k}]dt$$

$$\int\limits_c \mathbf{F} \times d\mathbf{r} = \mathbf{i}\int\limits_0^1 (-3t^5 - 2t^4)dt + \mathbf{j}\int\limits_0^1 (2t^5 - 6t^5)dt + \mathbf{k}\int\limits_0^1 (4t^3 + 2t^4)dt =$$

$$= -\frac{9}{10}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{5}\mathbf{k}.$$

Exercise 106

A force field is given by

$$\mathbf{F} = (y^2 \cos x + z^3)\mathbf{i} + (2y \sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}.$$

a) Prove that $\mathbf{F}$ is a conservative force.
b) Find the scalar potential ϕ of the force $\mathbf{F}$.
c) Find the work required to move a particle in this field from point $A(0,1,-1)$ to point $B(\pi/2,-1,2)$.

Solution

b) $\phi = y^2 \sin x + xz^3 - 4y + 2z + \text{const}$,   c) $15 + 4\pi$.

Exercise 107

Prove that $\mathbf{F} = r^2\mathbf{r}$ is a conservative force and find the scalar potential.

Solution

$\phi = \dfrac{r^4}{4} + \text{const}.$

Exercise 108

Let $\mathbf{E} = r\mathbf{r}$.

a) Is there a function ϕ that satisfies $\mathbf{E} = -\nabla\phi$? If there is, find it.
b) Calculate

$$\oint_c \mathbf{E} \cdot d\mathbf{r},$$

if c is any unambiguous closed curve.

Solution

a) $\phi = -\dfrac{r^3}{3} + \text{const}$,   b) 0.

Exercise 109

Show that

$$(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2 \sin y)\,dy + x\sin y\,dz,$$

is a total differential. From here, solve the differential equation

$$(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2 \sin y)\,dy + x\sin y\,dz = 0.$$

Solution

$x^2 \cos y + xz \sin y = \text{const.}$

Exercise 110

Let $\mathbf{F} = (x+2y)\mathbf{i} - 3z\mathbf{j} + x\mathbf{k}$, $\phi = 4x + 3y - 2z$, and S be the region of the surface $2x + y + 2z = 6$ bounded by the planes $x = 0$, $x = 1$, $y = 0$ and $y = 2$. Calculate the following integrals:

a)

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S.$$

b)

$$\iint_S \phi\, \mathbf{n}\, \mathrm{d}S.$$

Solution

a) 1, b) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Exercise 111

Let $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$, and V be a closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. Calculate

a)

$$\iiint_V (\nabla \cdot \mathbf{F})\mathrm{d}V,$$

b)

$$\iiint_V (\nabla \times \mathbf{F})\mathrm{d}V.$$

Solution

a) $8/3$, b) $\dfrac{8}{3}(\mathbf{j} - \mathbf{k})$.

Exercise 112

Calculate the integral

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\, \mathrm{d}x - 3x^2y^2\, \mathrm{d}y,$$

along the path $x^4 - 6xy^3 = 4y^2$.

Solution

It can be shown that $M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}x$ is a total differential if $\dfrac{\partial M}{\partial x}=\dfrac{\partial N}{\partial y}$[23]. In this Example $M=10x^4-2xy^3$ and $N=-3x^2y^2$, which satisfies the condition $\dfrac{\partial M}{\partial x}=\dfrac{\partial N}{\partial y}$, and it follows that $(10x^4-2xy^3)\,\mathrm{d}x-3x^2y^2\,\mathrm{d}y$ is the total differential of the function $\varphi=2x^5-x^2y^3$. From there, it follows that

$$\int_{(0,0)}^{(2,1)}(10x^4-2xy^3)\,\mathrm{d}x-3x^2y^2\,\mathrm{d}y=\int_{(0,0)}^{(2,1)}\mathrm{d}\varphi=$$
$$=\varphi\Big|_{(0,0)}^{(2,1)}=2x^5-x^2y^3\Big|_{(0,0)}^{(2,1)}=60.$$

Exercise 113

Show that the area bounded by a simple curved line C is given by

$$\frac{1}{2}\oint_C x\,\mathrm{d}y-y\,\mathrm{d}x.$$

Solution

According to Stokes' theorem (relation (4.90), p. 102)

$$\oint_C x\,\mathrm{d}y-y\,\mathrm{d}x=\iint_R\left(\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right)\mathrm{d}x\,\mathrm{d}y$$
$$=2\iint_R \mathrm{d}x\,\mathrm{d}y=2A,$$

where A is the required area. From here, it follows that

$$A=\frac{1}{2}\oint_C x\,\mathrm{d}y-y\,\mathrm{d}x.$$

23

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y=M(x,y)\mathrm{d}x+N(x,y)\mathrm{d}y,$$
$$N=\frac{\partial f}{\partial x},\quad M=\frac{\partial f}{\partial y}\quad\Rightarrow\quad\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial y}\quad \text{i}\quad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial M}{\partial x}\quad\Rightarrow$$
$$\frac{\partial N}{\partial y}=\frac{\partial M}{\partial x}.$$

Exercise 114

Calculate

$$\oint_C (y - \sin x)\,dx + \cos x\,dy,$$

where C is a triangle, depicted in Figure 4.31

a) directly,

b) by applying Stokes' theorem in the plane.

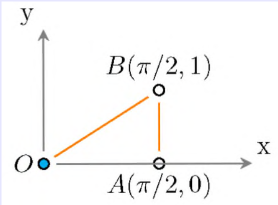


Figure 4.31: Path of integration.

Solution

a) Along OA, $y = 0$, $dy = 0$, and the integral is

$$\int_0^{\pi/2} (0 - \sin x)\,dx + \cos x\,(0) = \int_0^{\pi/2} -\sin x\,dx = \cos x\Big|_0^{\pi/2} = -1.$$

Along AB, $x = \pi/2$, $dx = 0$, and the integral is

$$\int_0^1 (y-1)(0) + 0\,dy = 0.$$

Along BO, $y = \dfrac{2x}{\pi}$, $dy = \dfrac{2}{\pi}dx$, and the integral is

$$\int_{\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right) dx + \frac{2}{\pi}\cos x\,dx = \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right)\Big|_{\pi/2}^{0} = 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

Consequently, the integral along the curve C is $= -1 + 0 + 1 - \dfrac{\pi}{4} - \dfrac{2}{\pi} = -\dfrac{\pi}{4} - \dfrac{2}{\pi}$.

b) According to Stokes' theorem (relation (4.90), p. 102)

$$\oint_c M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

As in our case $M = y - \sin x$, $N = \cos x$, $\dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = -\sin x$, we obtain

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-\sin x - 1)\,dx\,dy =$$

$$= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right] dx = \int_{x=0}^{\pi/2} (-y\sin x - y)\Big|_0^{2x/\pi} dx$$

$$= \int_{x=0}^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx = -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\Big|_0^{\pi/2} = -\frac{\pi}{4} - \frac{2}{\pi}.$$

As we can see, the result is the same as under a).

Exercise 115

Prove that

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = 0,$$

for any closed curve C in a simply connected region, iff $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ is valid in the entire region.

Proof

Assume that M and N are continuous functions with continuous partial derivatives in the entire region R, bounded by the curve C. Then, according to Stokes' theorem

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in the region R, then

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = 0.$$

Conversely, let

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = 0,$$

for each curve C.

Assume that there is at least one point P in R for which

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \neq 0.$$

Then this expression is different from zero also in some neighborhood A of point P, due to the continuity of the functions M and N. If the curve Γ is the boundary of region A then

$$0 \neq \oint_\Gamma M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y,$$

which is contrary to the assumption that the line integral is equal to zero for each closed curve. It thus follows that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ in all points of region R.

Exercise 116

Let $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

a) Calculate $\nabla \times \mathbf{F}$.
b) Calculate

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

for any closed path and explain the results.

Solution

a) For the given field $\mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \mathbf{0}$$

for all points (x, y)in the plane, except in $O(0,0)$. In point O the field $\mathbf{F}$ is not defined.

b) Observe the integral along the closed curve C

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}.$$

Let $x = \rho\cos\phi$, $y = \rho\sin\phi$, where (ρ, ϕ) are polar coordinates. From here, by differentiating, we obtain

$$\mathrm{d}x = -\rho\sin\phi\,\mathrm{d}\phi + \mathrm{d}\rho\cos\phi,$$
$$\mathrm{d}y = \rho\cos\phi\,\mathrm{d}\phi + \mathrm{d}\rho\sin\phi,$$

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$$\frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2} = \mathrm{d}\phi = \mathrm{d}\left(\arctan\frac{y}{x}\right).$$

We will consider two possible cases. The first, when the coordinate origin O is inside the curve $ABCDA$ (see Figure 4.26a), and the second, when the point O is outside the region surrounding this curve (Figure 4.26b). In the first case

$$\int_0^{2\pi} \mathrm{d}\phi = 2\pi,$$

and in the second

$$\int_{\phi_0}^{\phi_0} \mathrm{d}\phi = 0$$

(see Example on p. 141).

Exercise 117

Calculate $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$, where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$, and S is the surface of a cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

Solution

Using Gauss's theorem (see p. 102, relation (4.92)) we obtain

$$\iiint\limits_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint\limits_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right]\mathrm{d}V =$$

$$= \iiint\limits_V (4z - y)\,\mathrm{d}V = \int\limits_{x=0}^{1}\int\limits_{y=0}^{1}\int\limits_{z=0}^{1}(4z-y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x =$$

$$= \int\limits_{x=0}^{1}\int\limits_{y=0}^{1}(2z^2 - yz)\Big|_{z=0}^{1}\,\mathrm{d}y\,\mathrm{d}x =$$

$$= \int\limits_{x=0}^{1}\int\limits_{y=0}^{1}(2-y)\,\mathrm{d}y\,\mathrm{d}x = \frac{2}{3}.$$

Exercise 118

Calculate

$$I = \oint\limits_c \mathbf{A}\cdot\mathrm{d}\mathbf{r},$$

where $\mathbf{A} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, and c is the boundary of the surface S, the upper half of the sphere $x^2 + y^2 + z^2 = 1$:

a) directly,
b) using Stokes' theorem.

Solution

a) The boundary c of the surface S is a circle in the xy plane, with a radius of one, and center in the coordinate origin. If $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$ are parametric equation of the curve c, then

$$I = \oint\limits_c \mathbf{A}\cdot\mathrm{d}\mathbf{r} = \oint\limits_c [(2x-y)\mathrm{d}x - yz^2\mathrm{d}y - y^2z\mathrm{d}z] =$$

$$= \int\limits_0^{2\pi}(2\cos t - \sin t)(-\sin t)\mathrm{d}t = \pi.$$

b) According to Stokes' theorem

$$\oint\limits_c \mathbf{A}\cdot\mathrm{d}\mathbf{r} = \iint\limits_S (\nabla\times\mathbf{A})\cdot\mathrm{d}\mathbf{S} = I.$$

Given that

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \mathbf{k},$$

it follows that

$$I = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{k} \cdot \mathbf{n}\, dS = \iint_R dx\, dy.$$

Namely, $\mathbf{k} \cdot \mathbf{n}\, dS = dx\, dy$, R is the projection of S on the xy plane. By introducing these relations in the previous integral we obtain

$$I = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = 4 \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy =$$

$$= 4 \int_0^1 \sqrt{1 - x^2}\, dx = \pi.$$

Exercise 119

Prove that the necessary and sufficient condition for

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0,$$

along any closed curve C, is $\nabla \times \mathbf{A} = 0$.

Proof

The condition is sufficient.

In this case $\nabla \times \mathbf{A} = 0$ is valid. Then (Stokes' theorem 102, relation (4.92))

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = 0.$$

The condition is necessary.

In this case

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0, \tag{4.204}$$

for any closed curve C.

Further, according to Stokes' theorem,

$$0 = \oint_C \mathbf{A} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da$$

for any surface S passing through curve C. From here it follows that

$$\nabla \times \mathbf{A} = 0.$$

Exercise 120

Let ΔS be a surface bounded by a simple closed curve c, P any point on ΔS outside the curve c, $\mathbf{n}$ the unit vector of the normal to ΔS at P. Show that at point P the following is true

$$(\text{rot}\mathbf{A}) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{\oint\limits_c \mathbf{A} \cdot \mathrm{d}\mathbf{r}}{\Delta S},$$

where the boundary is taken in such a way that ΔS " tightens" around P.

Solution

According to Stokes' theorem

$$\iint\limits_{\Delta S} (\text{rot}\mathbf{A}) \cdot \mathbf{n}\, \mathrm{d}S = \oint\limits_c \mathbf{A} \cdot \mathrm{d}\mathbf{r}. \tag{4.205}$$

And according to the integral mean value theorem

$$\iint\limits_{\Delta S} (\text{rot}\mathbf{A}) \cdot \mathbf{n}\, \mathrm{d}S = \overline{(\text{rot}\mathbf{A}) \cdot \mathbf{n}}\Delta S. \tag{4.206}$$

From (4.205) and (4.206) it follows that

$$\overline{(\text{rot}\mathbf{A}) \cdot \mathbf{n}} = \frac{\oint\limits_c \mathbf{A} \cdot \mathrm{d}\mathbf{r}}{\Delta S},$$

that is

$$\lim_{\Delta S \to 0} \overline{(\text{rot}\mathbf{A}) \cdot \mathbf{n}} = (\text{rot}\mathbf{A}) \cdot \mathbf{n}|_P = \lim_{\Delta S \to 0} \frac{\oint\limits_c \mathbf{A} \cdot \mathrm{d}\mathbf{r}}{\Delta S},$$

which is the required result.

Exercise 121

If rot$\mathbf{A}$ is defined as in Example 120, find the projection of rot$\mathbf{A}$ on the z axis.

Solution

Let EFGH be a quadrangle parallel to the xy plane (its normal vector parallel to the z axis) with its center at point $P(x,y,z)$ (Figure 4.32). Let A_x and A_y be the projections of vector $\mathbf{A}$ on the x and y axes, respectively.

If c is the boundary of this quadrangle, then

$$\oint_c \mathbf{A}\cdot d\mathbf{r} = \int_{EF} \mathbf{A}\cdot d\mathbf{r} + \int_{FG} \mathbf{A}\cdot d\mathbf{r} + \int_{GH} \mathbf{A}\cdot d\mathbf{r} + \int_{HE} \mathbf{A}\cdot d\mathbf{r}.$$

The values of these integrals are

$$\int_{EF} \mathbf{A}\cdot d\mathbf{r} = \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial y}\Delta y\right)\Delta x$$
$$\int_{GH} \mathbf{A}\cdot d\mathbf{r} = -\left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial y}\Delta y\right)\Delta x$$
$$\int_{FG} \mathbf{A}\cdot d\mathbf{r} = \left(A_y + \frac{1}{2}\frac{\partial A_y}{\partial x}\Delta x\right)\Delta y$$
$$\int_{HE} \mathbf{A}\cdot d\mathbf{r} = -\left(A_y - \frac{1}{2}\frac{\partial A_y}{\partial x}\Delta x\right)\Delta y.$$

where all members of order higher than Δx, that is, Δy, are ignored,

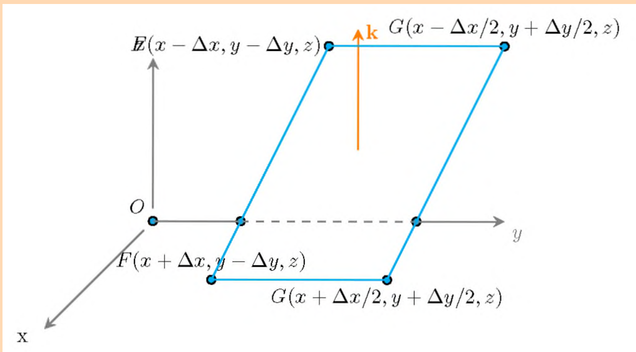

Figure 4.32

It follows from here that

$$\oint_c \mathbf{A}\cdot d\mathbf{r} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\Delta y.$$

As $\Delta S = \Delta x\Delta y$, the z projection of rot$\mathbf{A}$ is

$$(\text{rot}\mathbf{A})\cdot\mathbf{k} = \lim_{\Delta S\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{r}}{\Delta S} =$$
$$= \lim_{\substack{\Delta y\to 0\\ \Delta x\to 0}} \frac{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\Delta y}{\Delta x\Delta y} =$$
$$= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

Exercise 122

Calculate

$$\iiint\limits_V \nabla\cdot\mathbf{A}\,\mathrm{d}V,$$

where $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ is a vector field observed over the region V, bounded by surfaces $x^2 + y^2 = 4$, $z = 0$ and $z = 3$:

a) directly,

b) by using the divergence theorem (Gauss's theorem).

Solution

a) The volume V is the volume of a cylinder obtained when the unbounded cylinder $x^2 + y^2 = 4$ is intersected by planes $z = 0$ and $z = 3$, and hence the volume integral is

$$I = \iiint\limits_V \nabla\cdot\mathbf{A}\,\mathrm{d}V = \iiint\limits_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right]\mathrm{d}V =$$

$$= \iiint\limits_V (4 - 4y + 2z)\,\mathrm{d}V =$$

$$= \int\limits_{-2}^{2}\mathrm{d}x \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\mathrm{d}y \int\limits_0^3 (4 - 4y + 2z)\,\mathrm{d}z = 84\pi.$$

b) The volume V is bounded by the surface S, composed of the lower base S_1 ($z = 0$), upper base S_2 ($z = 3$) and side S_3 ($x^2 + y^2 = 4$). Applying the divergence theorem, we obtain the surface integral

$$I = I_1 + I_2 + I_3 = \iint\limits_S \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S =$$

$$= \iint\limits_{S_1} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S_1 + \iint\limits_{S_2} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S_2 + \iint\limits_{S_3} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S_3.$$

Let us now calculate the three integrals.
For S_1 ($z = 0$), $\mathbf{n} = -\mathbf{k}$, $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, and the integral is

$$I_1 = \iint\limits_{S_1} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S_1 = 0.$$

For S_2 ($z = 3$), $\mathbf{n} = \mathbf{k}$, $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, and the integral is

$$I_2 = \iint\limits_{S_2} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S_2 = 9\iint\limits_{S_2} \mathrm{d}S_2 = 36\pi,$$

because the area of S_2 is equal to 4π.
For S_3 ($x^2 + y^2 = 4$) the unit vector of the normal is

$$\mathbf{n} = \frac{\nabla(x^2 + y^2 - 4)}{|\nabla(x^2 + y^2 - 4)|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

$$\mathbf{A}\cdot\mathbf{n} = (4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k})\cdot\left(\frac{x\mathbf{i} + y\mathbf{j}}{2}\right) = 2x^2 - y^3.$$

It can be seen from Figure 4.33 theta $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2d\theta dz$, and it follows that

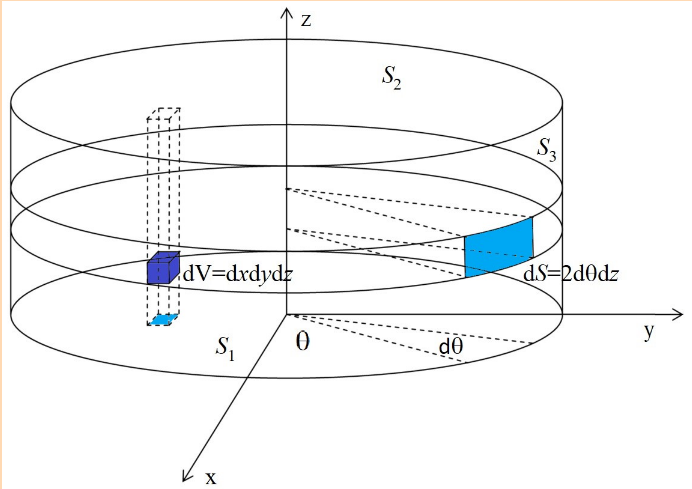

Figure 4.33

$$I_3 = \iint\limits_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int\limits_{\theta=0}^{2\pi}\int\limits_{z=0}^{3}[2(2\cos\theta)^2 - (2\sin\theta)^3)]2dz\,d\theta =$$

$$= \int\limits_{\theta=0}^{2\pi}(48\cos^2\theta - 48\sin^3\theta)\,d\theta =$$

$$= \int\limits_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi.$$

The surface integral over the entire surface is the sum of these three integrals, namely

$$I = I_1 + I_2 + I_3 = 0 + 36\pi + 48\pi = 84\pi.$$

Thus, the obtained result is the same as the one obtained by the volume integral, which confirms the divergence theorem.

Exercise 123

If div**A** represents the divergence of a vector field **A** at point P, show that

$$\text{div}\mathbf{A} = \lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}\mathbf{A}\cdot\mathbf{n}\,dS}{\Delta V},$$

where ΔV is the volume bounded by the closed contour of the surface ΔS, and the limit value is obtained by " shrinking" ΔV around point P.

Solution

According to Gauss's theorem (see relation (4.92) p. 102)

$$\iiint\limits_{\Delta V} \operatorname{div}\mathbf{A}\,\mathrm{d}V = \iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S. \tag{4.207}$$

and the mean value theorem ((4.93), p. 102)

$$\iiint\limits_{\Delta V} \operatorname{div}\mathbf{A}\,\mathrm{d}V = \overline{\operatorname{div}\mathbf{A}} \iiint\limits_{\Delta V} \mathrm{d}V = \overline{\operatorname{div}\mathbf{A}}\,\Delta V, \tag{4.208}$$

where $\overline{\operatorname{div}\mathbf{A}}$ is the mean value of $\operatorname{div}\mathbf{A}$ within ΔV, we obtain

$$\overline{\operatorname{div}\mathbf{A}} = \frac{1}{\Delta V} \iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S.$$

If we now let $\Delta V \to 0$, so that point P remains always within ΔV, then $\overline{\operatorname{div}\mathbf{A}}$ takes the value $\operatorname{div}\mathbf{A}$ at point P. Consequently

$$\operatorname{div}\mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint\limits_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S}{\Delta V}.$$

Exercise 124

Calculate the integral

$$\iint\limits_{S} \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}V,$$

where S is a closed surface.

Solution

According to the divergence theorem, we have

$$\iint\limits_{S} \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}S = \iiint\limits_{V} \nabla\cdot\mathbf{r}\,\mathrm{d}V =$$

$$= \iiint\limits_{V} \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,\mathrm{d}V =$$

$$= \iiint\limits_{V} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)\mathrm{d}V = 3\iiint\limits_{V} \mathrm{d}V = 3V,$$

where V is the volume bounded by the closed surface S.

Exercise 125

Prove that

$$\iiint\limits_V \nabla\phi \, \mathrm{d}V = \iint\limits_S \phi \mathbf{n} \, \mathrm{d}S.$$

Solution

Let $\mathbf{A} = \phi\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, and ϕ a scalar function. According to the divergence theorem

$$\iiint\limits_V \nabla(\phi\mathbf{C}) \, \mathrm{d}V = \iint\limits_S \phi\mathbf{C}\cdot\mathbf{n} \, \mathrm{d}S.$$

As (see Example 51b on p. 129) $\nabla\cdot(\phi\mathbf{C}) = (\nabla\phi)\cdot\mathbf{C} = \mathbf{C}\cdot(\nabla\phi)$ i $\phi\mathbf{C}\cdot\mathbf{n} = \mathbf{C}\cdot(\phi\mathbf{n})$, it follows that

$$\iiint\limits_V \mathbf{C}\cdot\nabla\phi \, \mathrm{d}V = \iint\limits_S \mathbf{C}\cdot(\phi\mathbf{n}) \, \mathrm{d}S,$$

that is

$$\mathbf{C}\cdot\iiint\limits_V \nabla\phi \, \mathrm{d}V = \mathbf{C}\cdot\iint\limits_S \phi\mathbf{n} \, \mathrm{d}S.$$

From here, given that $\mathbf{C}$ is an arbitrary vector, we finally obtain

$$\iiint\limits_V \nabla\phi \, \mathrm{d}V = \iint\limits_S \phi\mathbf{n} \, \mathrm{d}S.$$

Exercise 126

Prove that

$$\iiint\limits_V \nabla\times\mathbf{B} \, \mathrm{d}V = \iint\limits_S \mathbf{n}\times\mathbf{B} \, \mathrm{d}S.$$

Solution

Let $\mathbf{A} = \mathbf{B}\times\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector. According to the divergence theorem (Gauss's theorem (4.92))

$$\iiint\limits_V \nabla\cdot(\mathbf{B}\times\mathbf{C}) \, \mathrm{d}V = \iint\limits_S (\mathbf{B}\times\mathbf{C})\cdot\mathbf{n} \, \mathrm{d}S. \tag{4.209}$$

Based on the properties of the delta operator (see p. 88) and the properties of divergence (div $\mathbf{C}$=0), we obtain

$$\nabla\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\nabla\times\mathbf{B}). \tag{4.210}$$

On the other hand (see properties of the mixed product on p.26)

$$(\mathbf{B}\times\mathbf{C})\cdot\mathbf{n} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{n}) = (\mathbf{C}\times\mathbf{n})\cdot\mathbf{B} = \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B}). \tag{4.211}$$

Substituting the equations (4.210) and (4.211) in (4.209), we obtain

$$\iiint_V \mathbf{C}\cdot(\nabla\times\mathbf{B})\,\mathrm{d}V = \iint_S \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})\,\mathrm{d}S.$$

As $\mathbf{C}$ is aa arbitrary constant vector, it can be placed before the integral, and we finally obtain

$$\mathbf{C}\cdot\iiint_V \nabla\times\mathbf{B}\,\mathrm{d}V = \mathbf{C}\cdot\iint_S \mathbf{n}\times\mathbf{B}\,\mathrm{d}S \quad\Rightarrow$$
$$\iiint_V \nabla\times\mathbf{B}\,\mathrm{d}V = \iint_S \mathbf{n}\times\mathbf{B}\,\mathrm{d}S.$$

Exercise 127

Let P be a point within a body of volume ΔV, whose outer boundary is ΔS. Prove that at point P the following is true

a)

$$\nabla\phi = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\phi\mathbf{n}\,\mathrm{d}S}{\Delta V},$$

b)

$$\nabla\times\mathbf{A} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\mathbf{n}\times\mathbf{A}\,\mathrm{d}S}{\Delta V},$$

Solution

a) Given that (see Example 125, p. 183)

$$\iiint_V \nabla\phi\,\mathrm{d}V = \iint_{\Delta S}\phi\mathbf{n}\,\mathrm{d}S,$$

by a scalar product with $\mathbf{i}$ we obtain

$$\iiint_V \nabla\phi\cdot\mathbf{i}\,\mathrm{d}V = \iint_{\Delta S}\phi\mathbf{n}\cdot\mathbf{i}\,\mathrm{d}S.$$

Applying the mean value theorem yields

$$\overline{\nabla\phi\cdot\mathbf{i}} = \frac{\iint_{\Delta S}\phi\mathbf{n}\cdot\mathbf{i}\,\mathrm{d}S}{\Delta V},$$

where $\overline{\nabla\phi\cdot\mathbf{i}}$ is the mean value of $\nabla\phi\cdot\mathbf{i}$ in the entire ΔV. Taking the limit value, when $\Delta V\longrightarrow 0$, so that P remains within ΔV, we obtain

$$\nabla\phi\cdot\mathbf{i} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\phi\mathbf{n}\cdot\mathbf{i}\,\mathrm{d}S}{\Delta V}. \tag{4.212}$$

Analogously, we obtain

$$\nabla\phi\cdot\mathbf{j}=\lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}\phi\,\mathbf{n}\cdot\mathbf{j}\,\mathrm{d}S}{\Delta V} \tag{4.213}$$

$$\nabla\phi\cdot\mathbf{k}=\lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}\phi\,\mathbf{n}\cdot\mathbf{k}\,\mathrm{d}S}{\Delta V}. \tag{4.214}$$

If we now multiply the equations (4.212), (4.213) and (4.214), by **i**, **j**, **k**, respectively, and then add them, using

$$\nabla\phi=(\nabla\phi\cdot\mathbf{i})\mathbf{i}+(\nabla\phi\cdot\mathbf{j})\mathbf{j}+(\nabla\phi\cdot\mathbf{k})\mathbf{k}$$
$$\mathbf{n}=(\mathbf{n}\cdot\mathbf{i})\mathbf{i}+(\mathbf{n}\cdot\mathbf{j})\mathbf{j}+(\mathbf{n}\cdot\mathbf{k})\mathbf{k},$$

we obtain

$$\nabla\phi=\lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}\phi\,\mathbf{n}\,\mathrm{d}S}{\Delta V},$$

which was to be proved.

b) As (Example 126, p. 183)

$$\iiint\limits_{V}\nabla\times\mathbf{A}\,\mathrm{d}V=\iint\limits_{\Delta S}\mathbf{n}\times\mathbf{A}\,\mathrm{d}S,$$

similarly to the first part of this Example, by a scalar product with **i** we obtain

$$(\nabla\times\mathbf{A})\cdot\mathbf{i}=\lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}(\mathbf{n}\times\mathbf{A})\cdot\mathbf{i}\,\mathrm{d}S}{\Delta V}.$$

Analogously for **j** and **k**. Multiplying by **i**, **j** and **k**, and then adding, yields

$$\nabla\times\mathbf{A}=\lim_{\Delta V\to 0}\frac{\iint\limits_{\Delta S}\mathbf{n}\times\mathbf{A}\,\mathrm{d}S}{\Delta V}.$$

R Note that these results can be taken as starting points for defining the gradient, divergence and rotor. The expressions for gradient, divergence and rotor defined in this way are expressed independently of the coordinate system, and are thus valid for any coordinate system, i.e. they are invariant with respect to the coordinate system.

Exercise 128

Define the equivalency operator

$$\nabla\circ\equiv\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint\limits_{\Delta S}\mathrm{d}\mathbf{S}\,\circ,$$

where $\circ$ represents: multiplication of a vector by a scalar, the scalar product or the vector product, and the integral is calculated along a closed contour.

Solution

If $\circ$ represents the scalar product. then

$$\nabla \circ \mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathrm{d}\mathbf{S} \circ \mathbf{A}$$

or

$$\begin{aligned} \operatorname{div}\mathbf{A} &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathrm{d}\mathbf{S} \cdot \mathbf{A} = \\ &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\,\mathrm{d}S. \end{aligned}$$

Similarly, if $\circ$ represents the vector product, then (see Example 123)

$$\begin{aligned} \nabla \circ \mathbf{A} = \nabla \times \mathbf{A} = \operatorname{rot}\mathbf{A} &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathrm{d}\mathbf{S} \times \mathbf{A} = \\ &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{n} \times \mathbf{A}\,\mathrm{d}S. \end{aligned}$$

Finally, if $\circ$ represents the multiplication of a vector by a scalar ϕ, we obtain

$$\nabla \circ \phi = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathrm{d}\mathbf{S} \circ \phi,$$

or

$$\nabla \phi = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \phi\,\mathrm{d}\mathbf{S}.$$

(see Example 127a).

Exercise 129

Let S be a closed surface, V the space bounded by the surface S, and $\mathbf{r}$ the position vector of a point (x, y, z) with respect to the coordinate origin. Show that the integral

$$I = \iint_S \frac{\mathbf{n} \cdot \mathbf{r}}{r^3}\,\mathrm{d}S$$

a) $I = 0$, if O lies outside the surface S,
b) $I = 4\pi$ if O lies inside the surface S.

Solution

a) Using the divergence theorem we obtain

$$\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \iiint\limits_V \nabla\cdot\frac{\mathbf{r}}{r^3}\,\mathrm{d}V.$$

Given that $\nabla\cdot\frac{\mathbf{r}}{r^3} = 0$ (Example 53 on p. 131) it follows that

$$\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = 0.$$

b) If O is inside S, let us observe a sphere s with a radius a around the point O. Let τ be the region bounded by S and s. According to the divergence theorem

$$\iint\limits_{S+s} \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S + \iint\limits_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S =$$

$$= \iiint\limits_\tau \nabla\cdot\frac{\mathbf{r}}{r^3}\,\mathrm{d}V = 0,$$

because $r \neq 0$ in τ. From here it follows that

$$\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = -\iint\limits_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S.$$

For the sphere s, $r = a$, and $\mathbf{n} = -\frac{\mathbf{r}}{a}$ which yields

$$\frac{\mathbf{n}\cdot\mathbf{r}}{r^3} = \frac{-\frac{\mathbf{r}}{a}\cdot\mathbf{r}}{r^3} = -\frac{\mathbf{r}\cdot\mathbf{r}}{a^4} = -\frac{a^2}{a^4} = -\frac{1}{a^2}$$

and

$$\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = -\iint\limits_s \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,\mathrm{d}S = \iint\limits_s \frac{1}{a^2}\,\mathrm{d}S =$$

$$= \frac{1}{a^2}\iint\limits_s \mathrm{d}S = \frac{4\pi a^2}{a^2} = 4\pi.$$

4.6.7 Various examples

Exercise 130

Calculate

$$\oint\limits_c (3x+4y)\mathrm{d}x + (2x-3y)\mathrm{d}y,$$

where c is a circle with a radius of two, and center at the coordinate origin.

Solution

-8π.

Exercise 131

Calculate the itegral

$$\oint_c (x^2 - 2xy)\mathrm{d}x + (x^2y + 3)\mathrm{d}y,$$

if the boundary of the region is defined by the intersection of the lines $y^2 = 2x$ and $x = 2$:

a) directly,
b) using Green's theorem.

Solution

$128/5$.

Exercise 132

Calulate the integral

$$\int_c (6xy - y^2)\mathrm{d}x + (3x^2 - 2xy)\mathrm{d}y,$$

along the cycloid c, defined by $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, from point $A(0,0)$ to point $B(\pi,2)$.

Solution

$6\pi^2 - 4\pi$.

Exercise 133

Show that the surface

$$A = \iint_R \mathrm{d}x\mathrm{d}y,$$

after the transformation $x = x(u,v)$ and $y = y(u,v)$, is given by

$$A = \iint_R J\,\mathrm{d}u\mathrm{d}v, \qquad \text{gde je } J = \frac{\partial(x,y)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix},$$

J - Jacobian of the transformation.

Exercise 134

Calculate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k}$, and S is:

a) the parallelepiped bounded by $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 1$ i $z = 3$,

b) the surface bounding the region given by $x = 0$, $y = 0$, $z = 0$, $y = 3$ and $x + 2z = 6$.

Solution

a) 30, b) $351/2$.

Exercise 135

If $\mathbf{H} = \mathrm{rot}\mathbf{A}$, prove that

$$\iint_S \mathbf{H}\cdot\mathbf{n}\,dS = 0,$$

for a closed surface S.

Exercise 136

Prove that

$$\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}dS.$$

Exercise 137

Prove that

$$\iint_S r^5\mathbf{n}\,dS = \iiint_V 5r^3\mathbf{r}dV.$$

Exercise 138

Prove that

$$\iint_S \mathbf{n}\,dS = 0,$$

for any surface S.

Exercise 139

Calculate

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS,$$

where $\mathbf{A} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$, and the surface S is:
a) the half of the sphere $x^2 + y^2 + z^2 = 16$ above the xy plane,
b) the paraboloid $z = 4 - (x^2 + y^2)$ above the xy plane.

Solution

a) -16π, b) -4π.

Exercise 140

The potential $\phi(P)$, at point $P(x,y,z)$, in a system of charge particles $q_1, q_2, \ldots, q_n$ with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ is given by the formula

$$\phi = \sum_{m=1}^{n} \frac{q_m}{r_m}.$$

Prove Gauss's law

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi Q,$$

where $\mathbf{E} = -\nabla\phi$ is the strength of the electric field, S the surface encompassing all particles, and $Q = \sum_{m=1}^{n} q_m$ the total charge covered by S.

Exercise 141

Let V be a region bounded by S, ρ the fluid density, and $\phi(P)$ the potential at point P defined by

$$\phi = \iiint_V \frac{\rho \mathrm{d}V}{r}.$$

Prove that

a)

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi \iiint_V \rho \mathrm{d}V,$$

where $\mathbf{E} = -\nabla\phi$

b)

$$\nabla^2\phi = -4\pi\rho \quad \text{(Poisson's equation)},$$

at point P where there is fluid, and

$$\nabla^2\phi = 0 \quad \text{(Laplace's equation)},$$

where there is no fluid.

4.6.8 Generalised orthogonal systems

Curvilinear coordinates

Exercise 142

Determine the coordinate surfaces and coordinate lines for

a) cylindrical and
b) spherical coordinates.

Solution

a) The cylindrical coordinate system (ρ, φ, z).
The coordinate surfaces are
$\rho = c_1$ coaxial cylinders, with the centre of the base on the z axis,
$\varphi = c_2$ planes passing through the z axis,
$z = c_3$ planes normal to the z axis.
The coordinate lines are
the intersection of the surfaces $\rho = c_1$ and $\varphi = c_2$, yielding straight lines (z- axis),
the intersection of the surfaces $\rho = c_1$ and $z = c_3$, yielding circles,
the intersection of the surfaces $\varphi = c_2$ and $z = c_3$, yielding straight lines ($\rho > 0$).

b) The spherical coordinate system (r, θ, φ).
The coordinate surfaces are
$r = c_1$ concentric spheres, with the center on the z axis,
$\theta = c_2$ cone, with the vertex at the coordinate origin,
$\varphi = c_3$ planes, passing through the z axis.

The coordinate lines are
the intersection of the surfaces $r = c_1$ and $\theta = c_2$, yielding circles,
the intersection of the surfaces $r = c_1$ and $\varphi = c_3$, yielding semi-circles,
the intersection of the surfaces $\varphi = c_3$ and $\theta = c_2$, yielding lines ($r > 0$).

Exercise 143

Express the cylindrical coordinates in terms of Cartesian coordinates.

Solution

Let us start with the transformations expressing Cartesian coordinates in terms of cylindrical coordinates

$$x = \rho \cos \varphi \tag{4.215}$$

$$y = \rho \sin \varphi \tag{4.216}$$

$$z = z. \tag{4.217}$$

If we first square and then add equations (4.215) and (4.216), we obtain $\rho^2(\cos^2\varphi + \sin^2\varphi) = x^2 + y^2$, that is, $\rho = \sqrt{x^2+y^2}$. Here we used the basic trigonometric identity $\cos^2\varphi + \sin^2\varphi = 1$ and the fact that ρ, by definition (distance), is positive.

Dividing the left and right side of equation (4.216) by equation (4.215) yields $\frac{y}{x} = \frac{\rho\sin\varphi}{\rho\cos\varphi} = \tan\varphi$, that is, $\varphi = \arctan\frac{y}{x}$.

From here follow the transformations

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \varphi &= \arctan\frac{y}{x} \\ z &= z. \end{aligned}$$

Exercise 144

Show that the cylindric coordinate system is orthogonal.

Solution

The position vector is

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = \rho\cos\varphi\,\mathbf{i} + \rho\sin\varphi\,\mathbf{j} + z\,\mathbf{k}.$$

The tangent vectors, which correspond to coordinates ρ, φ, z are determined by $\frac{\partial\mathbf{r}}{\partial\rho}$, $\frac{\partial\mathbf{r}}{\partial\varphi}$ and $\frac{\partial\mathbf{r}}{\partial z}$, that is

$$\begin{aligned} \frac{\partial\mathbf{r}}{\partial\rho} &= \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}, \\ \frac{\partial\mathbf{r}}{\partial\varphi} &= -\rho\sin\varphi\,\mathbf{i} + \rho\cos\varphi\,\mathbf{j}, \\ \frac{\partial\mathbf{r}}{\partial z} &= \mathbf{k}. \end{aligned}$$

The corresponding unit vectors are

$$\begin{aligned} \mathbf{e}_1 = \mathbf{e}_\rho &= \frac{\partial\mathbf{r}/\partial\rho}{|\partial\mathbf{r}/\partial\rho|} = \frac{\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}, \\ \mathbf{e}_2 = \mathbf{e}_\varphi &= \frac{\partial\mathbf{r}/\partial\varphi}{|\partial\mathbf{r}/\partial\varphi|} = \frac{-\rho\sin\varphi\,\mathbf{i} + \rho\cos\varphi\,\mathbf{j}}{\sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi}} = -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j}, \\ \mathbf{e}_3 = \mathbf{e}_z &= \frac{\partial\mathbf{r}/\partial z}{|\partial\mathbf{r}/\partial z|} = \mathbf{k}. \end{aligned} \tag{4.218}$$

and thus

$$\begin{aligned} \mathbf{e}_1\cdot\mathbf{e}_2 &= (\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j})\cdot(-\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j}) = 0, \\ \mathbf{e}_1\cdot\mathbf{e}_3 &= (\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j})\cdot\mathbf{k} = 0, \\ \mathbf{e}_2\cdot\mathbf{e}_3 &= (-\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j})\cdot\mathbf{k} = 0. \end{aligned}$$

From here it is obvious that the unit vectors are normal to each other, namely, that the system is orthogonal.

Exercise 145

Express the vector $\mathbf{A} = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ in terms of cylindrical coordinates. Determine A_ρ, A_φ and A_z (projections of this vector on axes ρ, φ and z).

Solution

Vector $\mathbf{A}$ is expressed relative to the base $\mathbf{i}$, $\mathbf{j}$ i $\mathbf{k}$. In order to represent it in terms of cylindrical coordinates it is necessary to express the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in terms of the base $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$ and $\mathbf{e}_z$. These relations are given in the previous example (4.218):

$$\begin{aligned}\mathbf{e}_\rho &= \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j},\\ \mathbf{e}_\varphi &= -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j},\\ \mathbf{e}_z &= \mathbf{k},\end{aligned}$$

which yields

$$\begin{aligned}\mathbf{i} &= \cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi,\\ \mathbf{j} &= \sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi,\\ \mathbf{k} &= \mathbf{e}_z.\end{aligned} \tag{4.219}$$

According to the definition

$$\begin{aligned}A_\rho &= \mathbf{A}\cdot\mathbf{e}_\rho,\\ A_\varphi &= \mathbf{A}\cdot\mathbf{e}_\varphi,\\ A_z &= \mathbf{A}\cdot\mathbf{e}_z.\end{aligned} \tag{4.220}$$

By substituting (4.219) in the expression for vector $\mathbf{A}$, bearing in mind (4.220), we obtain

$$\begin{aligned}\mathbf{A} &= z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}\\ &= z(\cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi) - 2\rho\cos\varphi(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) + \rho\sin\varphi\,\mathbf{e}_z\\ &= (z\cos\varphi - 2\rho\cos\varphi\sin\varphi)\,\mathbf{e}_\rho - (z\sin\varphi + 2\rho\cos^2\varphi)\,\mathbf{e}_\varphi + \rho\sin\varphi\,\mathbf{e}_z,\end{aligned}$$

that is

$$\begin{aligned}A_\rho &= z\cos\varphi - 2\rho\cos\varphi\sin\varphi,\\ A_\varphi &= z\sin\varphi + 2\rho\cos^2\varphi,\\ A_z &= \rho\sin\varphi.\end{aligned}$$

Exercise 146

Prove that $\dfrac{\mathrm{d}\mathbf{e}_\rho}{\mathrm{d}t} = \dot{\varphi}\,\mathbf{e}_\varphi$, $\dfrac{\mathrm{d}\mathbf{e}_\varphi}{\mathrm{d}t} = -\dot{\varphi}\,\mathbf{e}_\rho$, where $\dot{\varphi} = \dfrac{\mathrm{d}\varphi}{\mathrm{d}t}$.

Solution

It was shown in Example 144 on p.192 (see (4.218)) that

$$\begin{aligned}\mathbf{e}_\rho &= \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j},\\ \mathbf{e}_\varphi &= -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j},\\ \mathbf{e}_z &= \mathbf{k}.\end{aligned}$$

By differentiating, we obtain from here

$$\begin{aligned}\frac{\mathrm{d}\mathbf{e}_\rho}{\mathrm{d}t} &= -\sin\varphi\,\dot{\varphi}\,\mathbf{i} + \cos\varphi\,\dot{\varphi}\,\mathbf{j} = (-\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j})\dot{\varphi} = \dot{\varphi}\,\mathbf{e}_\varphi,\\ \frac{\mathrm{d}\mathbf{e}_\varphi}{\mathrm{d}t} &= \cos\varphi\,\dot{\varphi}\,\mathbf{i} + \sin\varphi\dot{\varphi}\,\mathbf{j} \quad = (\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j})\dot{\varphi} \quad = -\dot{\varphi}\,\mathbf{e}_\rho,\end{aligned}$$

which was to be proved.

Exercise 147

Express the velocity **v** and acceleration **a** of a particle in cylindrical coordinates.

Solution

The position vector is $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and the velocity and acceleration are, by definition

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k},$$

that is

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}.$$

Let us now express the position vector in cylindrical coordinates

$$\begin{aligned}\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} &= (\rho\cos\varphi)(\cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi)\\ &\quad + (\rho\sin\varphi)(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) + z\,\mathbf{e}_z\\ &= \rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z.\end{aligned}$$

Differentiating by time we obtain the velocity

$$\begin{aligned}\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}\rho}{\mathrm{d}t}\,\mathbf{e}_\rho + \rho\,\frac{\mathrm{d}\mathbf{e}_\rho}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathbf{e}_z\\ &= \dot{\rho}\,\mathbf{e}_\rho + \rho\,\dot{\varphi}\,\mathbf{e}_\varphi + \dot{z}\,\mathbf{e}_z.\end{aligned}$$

Differentiating once again by time, we obtain the acceleration

$$\begin{aligned}\mathbf{a} &= \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{d}{\mathrm{d}t}(\dot{\rho}\,\mathbf{e}_\rho + \rho\dot{\varphi}\,\mathbf{e}_\varphi + \dot{z}\,\mathbf{e}_z)\\ &= \dot{\rho}\frac{\mathrm{d}\,\mathbf{e}_\rho}{\mathrm{d}t} + \ddot{\rho}\,\mathbf{e}_\rho + \rho\dot{\varphi}\frac{\mathrm{d}\,\mathbf{e}_\varphi}{\mathrm{d}t} + \rho\ddot{\varphi}\,\mathbf{e}_\varphi + \dot{\rho}\dot{\varphi}\,\mathbf{e}_\varphi + \ddot{z}\,\mathbf{e}_z\\ &= \dot{\rho}\dot{\varphi}\,\mathbf{e}_\rho + \ddot{\rho}\,\mathbf{e}_\rho + \rho\dot{\varphi}(-\dot{\varphi}\,\mathbf{e}_\rho) + \rho\ddot{\varphi}\,\mathbf{e}_\varphi + \dot{\rho}\dot{\varphi}\,\mathbf{e}_\varphi + \ddot{z}\,\mathbf{e}_z\\ &= (\ddot{\rho} - \rho\ddot{\varphi}^2)\,\mathbf{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\,\mathbf{e}_\varphi + \ddot{z}\,\mathbf{e}_z.\end{aligned}$$

Exercise 148

Find the infinitesimal part of an arc in cylindrical coordinates and determine the corresponding Lame's coefficients.

Solution

Given that

$$\begin{aligned}x &= \rho\cos\varphi\\ y &= \rho\sin\varphi\\ z &= z,\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}x &= -\rho\sin\varphi\,\mathrm{d}\varphi + \cos\varphi\,\mathrm{d}\rho,\\ \mathrm{d}y &= \rho\cos\varphi\,\mathrm{d}\varphi + \sin\varphi\,\mathrm{d}\rho,\\ \mathrm{d}z &= \mathrm{d}z,\end{aligned}$$

it follows that

$$\begin{aligned}(\mathrm{d}s)^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\\ &= (-\rho\sin\varphi\,\mathrm{d}\varphi + \cos\varphi\,\mathrm{d}\rho)^2 + (\rho\cos\varphi\,\mathrm{d}\varphi + \sin\varphi\,\mathrm{d}\rho)^2 + (\mathrm{d}z)^2\\ &= (\mathrm{d}\rho)^2 + \rho^2(\mathrm{d}\varphi)^2 + (\mathrm{d}z)^2 = h_1^2(\mathrm{d}\rho)^2 + h_2^2(\mathrm{d}\varphi)^2 + h_3^2(\mathrm{d}z)^2.\end{aligned}$$

From here, by comparison with (4.146) on p. 113, we obtain Lame's coefficients $h_1 = h_\rho = 1$, $h_2 = h_\varphi = \rho$ and $h_3 = h_z = 1$.

Exercise 149

Find the arc element in
  a) spherical and
  b) parabolic coordinates
and determine the corresponding Lame's coefficients.

Solution

a) The relation between Cartesian and spherical coordinates (r, θ, φ) are

$$x = r\sin\theta\cos\varphi,$$
$$y = r\sin\theta\sin\varphi,$$
$$z = r\cos\theta.$$

By differentiating we obtain

$$\mathrm{d}x = -r\sin\theta\sin\varphi\,\mathrm{d}\varphi + r\cos\theta\cos\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}r,$$
$$\mathrm{d}y = r\sin\theta\cos\varphi\,\mathrm{d}\varphi + r\cos\theta\sin\varphi\,\mathrm{d}\theta + \sin\theta\sin\varphi\,\mathrm{d}r,$$
$$\mathrm{d}z = -r\sin\theta\,\mathrm{d}\theta + \cos\theta\,\mathrm{d}r,$$

and thus the square of the arc element is

$$(\mathrm{d}s)^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\varphi)^2.$$

From here we obtain Lame's coefficients $h_1 = h_r = 1$, $h_2 = h_\theta = r$ and $h_3 = h_\varphi = r\sin\theta$.

b) The relation between Cartesian and parabolic coordinates (u, v, z) are

$$x = \frac{1}{2}(u^2 - v^2),$$
$$y = uv,$$
$$z = z.$$

By differentiating we obtain

$$\mathrm{d}x = u\,\mathrm{d}u - v\,\mathrm{d}v,$$
$$\mathrm{d}y = u\,\mathrm{d}v + v\,\mathrm{d}u,$$
$$\mathrm{d}z = \mathrm{d}z.$$

The square of the arc element is

$$(\mathrm{d}s)^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = (u^2 + v^2)(\mathrm{d}u)^2 + (u^2 + v^2)(\mathrm{d}v)^2 + (\mathrm{d}z)^2,$$

and hence the Lame's coefficients are $h_1 = h_u = \sqrt{u^2 + v^2}$, $h_2 = h_v = \sqrt{u^2 + v^2}$ and $h_3 = h_z = 1$.

Exercise 150

Determine the infinitesimal part of a volume $\mathrm{d}V$ in

a) cylindrical,
b) spherical coordinates.

Solution

a) From Figure 4.22 we can see that the sides of the shaded body are $\rho\,\mathrm{d}\varphi$, $\mathrm{d}\rho$ and $\mathrm{d}z$. Given that the system of cylindrical coordinates is orthogonal, it follows that the elementary volume is $\mathrm{d}V = \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3$ (see p. 113, relation (4.147)). As the lengths of the sides are

$$\begin{aligned} \mathrm{d}s_1 &= h_1\,\mathrm{d}u^1 = 1\cdot(\mathrm{d}\rho) = \mathrm{d}\rho, \\ \mathrm{d}s_2 &= h_2\,\mathrm{d}u^2 = \rho\cdot(\mathrm{d}\varphi) = \rho\,\mathrm{d}\varphi, \\ \mathrm{d}s_3 &= h_3\,\mathrm{d}u^3 = 1\cdot(\mathrm{d}z) = \mathrm{d}z, \end{aligned}$$

the volume is

$$\mathrm{d}V = h_1h_2h_3\,\mathrm{d}u^1\,\mathrm{d}u^2\,\mathrm{d}u^3 \quad \Rightarrow$$

or

$$\mathrm{d}V = 1\cdot\rho\cdot 1\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z.$$

b) From Figure 4.23 we can see that the sides of the shaded body are $\mathrm{d}r$, $r\mathrm{d}\theta$ and $r\sin\theta\mathrm{d}\varphi$. Given that the system of spherical coordinates is orthogonal, it follows that the elementary volume is $\mathrm{d}V = \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3$. As the lengths of the sides are

$$\begin{aligned} \mathrm{d}s_1 &= h_1\,\mathrm{d}u^1 = 1\cdot(\mathrm{d}r) = \mathrm{d}r, \\ \mathrm{d}s_2 &= h_2\,\mathrm{d}u^2 = r\cdot(\mathrm{d}\theta) = r\,\mathrm{d}\theta, \\ \mathrm{d}s_3 &= h_3\,\mathrm{d}u^3 = r\sin\theta\cdot(\mathrm{d}\varphi) = r\sin\theta\,\mathrm{d}\varphi, \end{aligned}$$

the volume is

$$\mathrm{d}V = h_1h_2h_3\,\mathrm{d}u^1\,\mathrm{d}u^2\,\mathrm{d}u^3,$$

or

$$\mathrm{d}V = 1\cdot r\cdot r\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\varphi = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\varphi.$$

Exercise 151

Find Lame's coefficients and the volume element $\mathrm{d}V$ in spheroidal coordinates.

Solution

The relation between the Cartesian and the spheroidal coordinate systems is

$$\begin{aligned} x &= a\,\mathrm{ch}\,\xi\cos\eta\cos\varphi, \\ y &= a\,\mathrm{ch}\,\xi\cos\eta\sin\varphi, \\ z &= a\,\mathrm{sh}\,\xi\sin\eta. \end{aligned} \tag{4.221}$$

The procedure for determining Lame's coefficients and calculating the volume element is the same as for the previously observed coordinate systems: calculating $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, and then determining $\mathrm{d}s$ and Lame's coefficients, and finally determining the volume element.

By differentiating (4.221) we obtain

$$\begin{aligned}
\mathrm{d}x &= -a\,\mathrm{ch}\,\xi\cos\eta\sin\varphi\,\mathrm{d}\varphi - a\,\mathrm{ch}\,\xi\sin\eta\cos\varphi\,\mathrm{d}\eta + a\,\mathrm{sh}\,\xi\cos\eta\cos\varphi\,\mathrm{d}\xi\,,\\
\mathrm{d}y &= a\,\mathrm{ch}\,\xi\cos\eta\cos\varphi\,\mathrm{d}\varphi - a\,\mathrm{ch}\,\xi\sin\eta\sin\varphi\,\mathrm{d}\eta + a\,\mathrm{sh}\,\xi\cos\eta\sin\varphi\,\mathrm{d}\xi\,,\\
\mathrm{d}z &= a\,\mathrm{sh}\,\xi\cos\eta\,\mathrm{d}\eta + a\,\mathrm{ch}\,\xi\sin\eta\,\mathrm{d}\xi\,.
\end{aligned}$$

The square of the arc element is

$$\begin{aligned}
(\mathrm{d}s)^2 =& \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 = a^2(\mathrm{sh}^2\,\xi+\sin^2\eta)(\mathrm{d}\xi)^2\\
&+a^2(\mathrm{sh}^2\,\xi+\sin^2\eta)(\mathrm{d}\eta)^2 + a\,\mathrm{ch}^2\,\xi\cos^2\eta(\mathrm{d}\varphi)^2. \quad \Rightarrow
\end{aligned}$$

Lame's coefficients are

$$\begin{aligned}
h_1 &= h_\xi = a\sqrt{\mathrm{sh}^2\,\xi+\sin^2\eta}\,,\\
h_2 &= h_\eta = a\sqrt{\mathrm{sh}^2\,\xi+\sin^2\eta}\,,\\
h_3 &= h_\varphi = a\,\mathrm{ch}\,\xi\cos\eta\,.
\end{aligned}$$

The elementary volume is

$$\begin{aligned}
\mathrm{d}V =& \left(a\sqrt{\mathrm{sh}^2\,\xi+\sin^2\eta}\right)\left(a\sqrt{\mathrm{sh}^2\,\xi+\sin^2\eta}\right)(a\,\mathrm{ch}\,\xi\cos\eta)\,\mathrm{d}\varphi\,\mathrm{d}\eta\,\mathrm{d}\xi =\\
&= a^3\left(\mathrm{sh}^2\,\xi+\sin^2\eta\right)\mathrm{d}\varphi\,\mathrm{d}\eta\,\mathrm{d}\xi\,.
\end{aligned}$$

Exercise 152

Express the elementary area in terms of generalized coordinates.

Solution

We have shown that the differential of the position vector can be expressed in generalized coordinates as follows (see relation (4.138) on p. 112)

$$\mathrm{d}\mathbf{r} = h_1\mathrm{d}u^1\mathbf{e}_1 + h_2\mathrm{d}u^2\mathbf{e}_2 + h_3\mathrm{d}u^3\mathbf{e}_3 = \sum_{i=1}^{3} h_i\mathrm{d}u^i\mathbf{e}_i.$$

In particular, along the coordinate line u^1 the coordinates u^2 and u^3 are constant, and it follows that $\mathrm{d}\mathbf{r} = h_1\mathrm{d}u^1\mathbf{e}_1$. The length of the arc element $\mathrm{d}s_1$, along the u^1 coordinate line, at point P, is

$$\mathrm{d}s_1 = h_1\mathrm{d}u^1.$$

Similarly, we obtain the expressions for $\mathrm{d}s_2$ and $\mathrm{d}s_3$ along the coordinate lines u^2 and u^3, respectively.

As the area can be expressed in terms of a vector product (see Example 10 on p.

62), the area formed by the two lengths $\mathrm{d}s_1$ and $\mathrm{d}s_2$ is given by

$$\begin{aligned}\mathrm{d}A_1 &= \left|(h_2\mathrm{d}u^2\mathbf{e}_2)\times(h_3\mathrm{d}u^3\mathbf{e}_3)\right| = \\ &= h_2h_3\mathrm{d}u^2\mathrm{d}u^3\,|\mathbf{e}_2\times\mathbf{e}_3| = h_2h_3\mathrm{d}u^2\mathrm{d}u^3\,|\mathbf{e}_1| = \\ &= h_2h_3\mathrm{d}u^2\mathrm{d}u^3.\end{aligned}$$

Similarly, for the remaining two areas we obtain

$$\begin{aligned}\mathrm{d}A_2 &= h_3h_1\mathrm{d}u^3\mathrm{d}u^1,\\ \mathrm{d}A_3 &= h_1h_2\mathrm{d}u^1\mathrm{d}u^2,\end{aligned}$$

or more shortly expressed

$$\mathrm{d}A_i = \sum_{j,k=1}^{3} e_{ijk}h_jh_k\mathrm{d}u^j\mathrm{d}u^k,$$

where e_{ijk} is the alternation tensor, defined by

$$e_{ijk} = \begin{cases} e_{123} = +1; \\ +1, \text{ if } ijk \text{ is an even permutation of indices 1,2,3,} \\ -1, \text{ if } ijk \text{ is an odd permutation of indices 1,2,3,} \\ \quad 0, \text{ in all other cases.}\end{cases}$$

Exercise 153

Let u^1, u^2 and u^3 be the generalized orthogonal coordinates. Prove that the Jacobian of the transformation, symbolically denoted in one of the two following forms

$$J = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)} = \begin{vmatrix} \dfrac{\partial x}{\partial u^1} & \dfrac{\partial y}{\partial u^1} & \dfrac{\partial z}{\partial u^1} \\ \dfrac{\partial x}{\partial u^2} & \dfrac{\partial y}{\partial u^2} & \dfrac{\partial z}{\partial u^3} \\ \dfrac{\partial x}{\partial u^3} & \dfrac{\partial y}{\partial u^3} & \dfrac{\partial z}{\partial u^3} \end{vmatrix}$$

is equal to

$$J = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)} = h_1h_2h_3.$$

Solution

If we introduce new variables u^1, u^2 and u^3 instead of Cartesian coordinates x, y and z

by the following relations

$$x = x(u^1, u^2, u^3),$$
$$y = y(u^1, u^2, u^3),$$
$$z = z(u^1, u^2, u^3),$$

where $x(u^i)$, $y(u^i)$ and $z(u^i)$, are continuous functions differentiable by u^i, $i = 1,2,3$, in a region V, then the total differentials are

$$\mathrm{d}x = \frac{\partial x}{\partial u^1}\mathrm{d}u^1 + \frac{\partial x}{\partial u^2}\mathrm{d}u^2 + \frac{\partial x}{\partial u^3}\mathrm{d}u^3,$$
$$\mathrm{d}y = \frac{\partial y}{\partial u^1}\mathrm{d}u^1 + \frac{\partial y}{\partial u^2}\mathrm{d}u^2 + \frac{\partial y}{\partial u^3}\mathrm{d}u^3,$$
$$\mathrm{d}z = \frac{\partial z}{\partial u^1}\mathrm{d}u^1 + \frac{\partial z}{\partial u^2}\mathrm{d}u^2 + \frac{\partial z}{\partial u^3}\mathrm{d}u^3,$$

or in matrix form

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{bmatrix} \begin{bmatrix} \mathrm{d}u^1 \\ \mathrm{d}u^2 \\ \mathrm{d}u^3 \end{bmatrix}.$$

The square matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{bmatrix}$$

is the variable transformation matrix. Its determinant is the so called functional determinant or Jacobian,, symbolically denoted by

$$J = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)}.$$

The elements of this determinant can be related to the tangent base vectors of the coordinate axes

$$\frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial}{\partial u^i}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial u^i}\mathbf{i} + \frac{\partial y}{\partial u^i}\mathbf{j} + \frac{\partial z}{\partial u^i}\mathbf{k}.$$

As a mixed product can be expressed in terms of a formal determinant (see Example 15, p. 64), it follows that the functional determinant is equal to the mixed product

$$\begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} = \frac{\partial \mathbf{r}}{\partial u^1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3} \right). \qquad (4.222)$$

These vectors can be expressed in terms of Lame's coefficients h_i and unit vectors $\mathbf{e}_i$ (see p. 110, equation (4.133)), and it follows that the value of the determinant is (4.222)

$$\begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} = h_1\mathbf{e}_1 \cdot (h_2\mathbf{e}_2 \times h_3\mathbf{e}_3) = h_1\, h_2\, h_3\, \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)\,.$$

As $\mathbf{e}_i$ are orthonormalized vectors, it follows that $\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = 1$, and we finally obtain

$$J = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} = h_1\, h_2\, h_3.$$

R Note. if the Jacobian is identically equal to zero then $\frac{\partial \mathbf{r}}{\partial u^1}$, $\frac{\partial \mathbf{r}}{\partial u^2}$ and $\frac{\partial \mathbf{r}}{\partial u^3}$ are coplanar vectors, namely, they lie in one plane and are linearly dependent. Thus, in that case, x, y and z are not independent, that is, there exists a function in the form $F(x,y,z) = 0$. The opposite is also true. Thus, $J \neq 0$ is the necessary and sufficient condition for the following coordinate transformation to exist

$$\begin{aligned} x &= x(u^1,u^2,u^3), \\ y &= y(u^1,u^2,u^3), \\ z &= z(u^1,u^2,u^3) \end{aligned}$$

as well as its inverse transformation

$$u^i = u^i(x,y,z), \qquad i = 1,2,3.$$

Exercise 154

Let u^1, u^2 and u^3 be the generalized curvilinear coordinates. Prove that $\frac{\partial \mathbf{r}}{\partial u^1}$, $\frac{\partial \mathbf{r}}{\partial u^2}$, $\frac{\partial \mathbf{r}}{\partial u^3}$ and ∇u^1, ∇u^2, ∇u^3 are reciprocal vectors.

Solution

The necessary and sufficient condition for the vectors to be reciprocal is

$$\frac{\partial \mathbf{r}}{\partial u^p} \cdot \nabla u^q = \begin{cases} 1, & \text{for} \quad p = q, \\ 0, & \text{for} \quad p \neq q, \end{cases} \quad p,q = 1,2,3.$$

In this case

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^1}\mathrm{d}u^1 + \frac{\partial \mathbf{r}}{\partial u^2}\mathrm{d}u^2 + \frac{\partial \mathbf{r}}{\partial u^3}\mathrm{d}u^3,$$

and by a scalar product with ∇u^1, we obtain

$$\nabla u_1 \cdot \mathrm{d}\mathbf{r} = \mathrm{d}u_1 = \left(\nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_1}\right) \mathrm{d}u_1 + \left(\nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_2}\right) \mathrm{d}u_2 + \left(\nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_3}\right) \mathrm{d}u_3,$$

or

$$\nabla u^1 \cdot \frac{\partial \mathbf{r}}{\partial u^1} = 1, \qquad \nabla u^1 \cdot \frac{\partial \mathbf{r}}{\partial u^2} = 0, \qquad \nabla u^1 \cdot \frac{\partial \mathbf{r}}{\partial u^3} = 0.$$

Similarly, it can be shown that

$$\nabla u^2 \cdot \frac{\partial \mathbf{r}}{\partial u^p} = \delta_{2p} \quad \text{and} \quad \nabla u^3 \cdot \frac{\partial \mathbf{r}}{\partial u^p} = \delta_{3p}.$$

Exercise 155

Prove that

$$\left[\frac{\partial \mathbf{r}}{\partial u^1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3}\right)\right] \left[\nabla u^1 \cdot \left(\nabla u^2 \times \nabla u^3\right)\right] = 1.$$

Solution

In the previous example we have shown that $\frac{\partial \mathbf{r}}{\partial u^1}, \frac{\partial \mathbf{r}}{\partial u^2}, \frac{\partial \mathbf{r}}{\partial u^3}$ and ∇u^1, ∇u^2, ∇u^3 are reciprocal vectors.

Let us denote the corresponding Jacobians by

$$J = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)} \quad \text{and} \quad j = \frac{\partial(u^1,u^2,u^3)}{\partial(x,y,z)}.$$

The Jacobians are equal to corresponding mixed products (see Example 153, p. 199)

$$J = \left|\left[\frac{\partial \mathbf{r}}{\partial u^1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3}\right)\right]\right|,$$
$$j = \left|\left[\nabla u^1 \cdot \left(\nabla u^2 \times \nabla u^3\right)\right]\right|.$$

Further, according to the theorem that states that a mixed product of three vectors yields a volume, and the mixed product of the reciprocal vectors yields the reciprocal volume (see Example 20c, on p. 66), we obtain

$$\nabla u^1 \cdot \left(\nabla u^2 \times \nabla u^3\right) = \begin{vmatrix} \frac{\partial u^1}{\partial x} & \frac{\partial u^1}{\partial y} & \frac{\partial u^1}{\partial z} \\ \frac{\partial u^2}{\partial x} & \frac{\partial u^2}{\partial y} & \frac{\partial u^3}{\partial z} \\ \frac{\partial u^3}{\partial x} & \frac{\partial u^3}{\partial y} & \frac{\partial u^3}{\partial z} \end{vmatrix} = j$$

and

$$\frac{1}{\mathrm{d}V}\mathrm{d}V = j \cdot J = 1,$$

or

$$\left[\frac{\partial \mathbf{r}}{\partial u^1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3}\right)\right] \left[\nabla u^1 \cdot \left(\nabla u^2 \times \nabla u^3\right)\right] = 1,$$

which was to be proved.

Exercise 156

Prove that the square of the arc element, in generalized coordinates, can be expressed as follows

$$\mathrm{d}s^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq} \mathrm{d}u^p \, \mathrm{d}u^q.$$

Solution

We have

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^1}\mathrm{d}u^1 + \frac{\partial \mathbf{r}}{\partial u^2}\mathrm{d}u^2 + \frac{\partial \mathbf{r}}{\partial u^3}\mathrm{d}u^3 = \alpha_1 \mathrm{d}u^1 + \alpha_2 \mathrm{d}u^2 + \alpha_3 \mathrm{d}u^3,$$

so that

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} =& \alpha_1 \cdot \alpha_1 (\mathrm{d}u^1)^2 + \alpha_1 \cdot \alpha_2 \mathrm{d}u^1 \mathrm{d}u^2 + \alpha_1 \cdot \alpha_3 \mathrm{d}u^1 \mathrm{d}u^3 \\ &+ \alpha_2 \cdot \alpha_1 \mathrm{d}u^2 \mathrm{d}u^1 + \alpha_2 \cdot \alpha_2 (\mathrm{d}u^2)^2 + \alpha_2 \cdot \alpha_3 \mathrm{d}u^2 \mathrm{d}u^3 \\ &+ \alpha_3 \cdot \alpha_1 \mathrm{d}u^3 \mathrm{d}u^1 + \alpha_3 \cdot \alpha_2 \mathrm{d}u^3 \mathrm{d}u^2 + \alpha_3 \cdot \alpha_3 (\mathrm{d}u^3)^2 \\ &= \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq} \mathrm{d}u^p \, \mathrm{d}u^q \qquad \text{gde je } g_{pq} = \alpha_p \cdot \alpha_q.\end{aligned}$$

This expression is called the *fundamental quadratic form* or *metric form*. The values g_{pq} are called *metric coefficients* and are symmetrical ($g_{pq} = g_{qp}$). If $g_{pq} = 0$ for $p \neq q$ then the coordinate system is orthogonal. Then $g_{11} = h_1^2$, $g_{22} = h_2^2$, $g_{33} = h_3^2$.

4.6.9 Gradient, divergence and rotor in generalized orthogonal coordinates

Exercise 157

Determine $\nabla\phi$ in generalized orthogonal coordinates.

Solution

Let

$$\nabla\phi = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3, \tag{4.223}$$

where f_i, $i = 1,2,3$ are unambiguous functions of u^i. Given that

$$\begin{aligned} \mathrm{d}\mathbf{r} =& \frac{\partial \mathbf{r}}{\partial u^1}\mathrm{d}u^1 + \frac{\partial \mathbf{r}}{\partial u^2}\mathrm{d}u^2 + \frac{\partial \mathbf{r}}{\partial u^3}\mathrm{d}u^3 = \\ =& h_1\mathbf{e}_1\mathrm{d}u^1 + h_2\mathbf{e}_2\mathrm{d}u^2 + h_3\mathbf{e}_3\mathrm{d}u^3, \end{aligned}$$

it follows (see p. 85, (4.26))

$$\mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\mathbf{r} = h_1 f_1 \mathrm{d}u^1 + h_2 f_2 \mathrm{d}u^2 + h_3 f_3 \mathrm{d}u^3. \tag{4.224}$$

On the other hand, the total differential of the scalar function $\phi(u^i)$ is

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial u^1}\mathrm{d}u^1 + \frac{\partial \phi}{\partial u^2}\mathrm{d}u^2 + \frac{\partial \phi}{\partial u^3}\mathrm{d}u^3. \tag{4.225}$$

From (4.224) and (4.225) it follows that

$$f_1 = \frac{1}{h_1}\frac{\partial \phi}{\partial u^1}, \qquad f_2 = \frac{1}{h_2}\frac{\partial \phi}{\partial u^2}, \qquad f_3 = \frac{1}{h_3}\frac{\partial \phi}{\partial u^3}. \tag{4.226}$$

Substituting the values (4.226) into (4.223) we obtain

$$\begin{aligned} \nabla\phi =& \frac{\mathbf{e}_1}{h_1}\frac{\partial \phi}{\partial u^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial \phi}{\partial u^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial \phi}{\partial u^3} = \\ =& \left(\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u^3}\right)\phi. \end{aligned}$$

Thus, the ∇ operator, in orthogonal coordinates, is

$$\nabla \equiv \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u^3}.$$

Exercise 158

Let u^1, u^2 and u^3 be orthogonal coordinates. Prove that

a) $|\nabla u_p| = h_p^{-1}$,

b) $\mathbf{e}_p = \mathbf{E}_p$, $\qquad p = 1,2,3.$

Solution

a) Introducing the substitution $\phi = u^1$ in relations on page 203, we obtain $\nabla u^1 = \dfrac{\mathbf{e}_1}{h_1}$. The magnitude of this vector is $|\nabla u^1| = \dfrac{|\mathbf{e}_1|}{h_1} = h_1^{-1}$, because $|\mathbf{e}_1| = 1$. Repeating this procedure for $\phi = u^2$ and $\phi = u^3$ yields

$$|\nabla u^2| = \frac{|\mathbf{e}_2|}{h_2} = h_2^{-1}$$

$$|\nabla u^3| = \frac{|\mathbf{e}_3|}{h_3} = h_3^{-1},$$

or shortly

$$|\nabla u^p| = h_p^{-1}, \qquad p = 1,2,3.$$

b) According to the definition $\mathbf{E}_p = \dfrac{\nabla u^p}{|\nabla u^p|}$. Using the result from this example under a) we obtain

$$\mathbf{E}_p = h_p \nabla u^p = \mathbf{e}_p,$$

what was to be proved.

Exercise 159

Prove that $\mathbf{e}_1 = h_2 h_3 \nabla u^2 \times \nabla u^3$ and similarly for $\mathbf{e}_2$ and $\mathbf{e}_3$, where u^1, u^2 and u^3 are orthogonal coordinates.

Solution

From previous example we have

$$\nabla u^1 = \frac{\mathbf{e}_1}{h_1}, \qquad \nabla u^2 = \frac{\mathbf{e}_2}{h_2}, \qquad \nabla u^3 = \frac{\mathbf{e}_3}{h_3},$$

and thus

$$\nabla u^2 \times \nabla u^3 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{h_2 h_3} = \frac{\mathbf{e}_1}{h_2 h_3} \qquad \text{and} \qquad \mathbf{e}_1 = h_2 h_3 \nabla u^2 \times \nabla u^3.$$

In the same way, for the remaining two vectors we obtain

$$\mathbf{e}_2 = h_3 h_1 \nabla u^3 \times \nabla u^1 \qquad \text{and} \qquad \mathbf{e}_3 = h_1 h_2 \nabla u^1 \times \nabla u^2,$$

what was to be proved.

Exercise 160

Show that the following is true for orthogonal coordinates

a) $$\nabla \cdot (A_1 \mathbf{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u^1}(A_1 h_2 h_3)$$

b) $$\nabla \times (A_1 \mathbf{e}_1) = \frac{\mathbf{e}_2}{h_3 h_1} \frac{\partial}{\partial u^3}(A_1 h_1) - \frac{\mathbf{e}_3}{h_1 h_2} \frac{\partial}{\partial u^2}(A_1 h_1),$$

as well as that analogous relations are true for vectors $A_2 \mathbf{e}_2$ and $A_3 \mathbf{e}_3$.

Solution

a) Using the expressions for $\mathbf{e}_i$ from Example 159, p. 205 we obtain

$$\begin{aligned}\nabla\cdot(A_1\mathbf{e}_1) =& \nabla\cdot(A_1h_2h_3\nabla u^2\times\nabla u^3) =\\ =&\nabla(A_1h_2h_3)\cdot\nabla u^2\times\nabla u^3 + A_1h_2h_3\nabla\cdot(\nabla u^2\times\nabla u^3) =\\ =&\nabla(A_1h_2h_3)\cdot\frac{\mathbf{e}_2}{h_2}\times\frac{\mathbf{e}_3}{h_3}+0 = \nabla(A_1h_2h_3)\cdot\frac{\mathbf{e}_1}{h_2h_3} =\\ =&\left[\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u^2}(A_1h_2h_3)+\frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u^3}(A_1h_2h_3)\right]\cdot\frac{\mathbf{e}_1}{h_2h_3} =\\ =&\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u^1}(A_1h_2h_3).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\nabla\cdot(A_2\mathbf{e}_2) =& \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}(A_2h_3h_1),\\ \nabla\cdot(A_3\mathbf{e}_3) =& \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(A_3h_1h_2).\end{aligned}$$

b) Using the expressions for $\mathbf{e}_i$ from Example 159 we obtain

$$\begin{aligned}\nabla\times(A_1\mathbf{e}_1) =& \nabla\times(A_1h_1\nabla u^1) =\\ =&\nabla(A_1h_1)\times\nabla u^1 + A_1h_1\nabla\times\nabla u^1 =\\ =&\nabla(A_1h_1)\times\frac{\mathbf{e}_1}{h_1}+0 =\\ =&\left[\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u^1}(A_1h_1)+\frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u^2}(A_1h_1)+\frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u^3}(A_1h_1)\right]\times\frac{\mathbf{e}_1}{h_1} =\\ =&\frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u^3}(A_1h_1)-\frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u^2}(A_1h_1).\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}\nabla\times(A_2\mathbf{e}_2) =& \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u^1}(A_2h_2)-\frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u^3}(A_2h_2),\\ \nabla\times(A_3\mathbf{e}_3) =& \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u^2}(A_3h_3)-\frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u^1}(A_3h_3).\end{aligned}$$

Exercise 161

Express rot$\mathbf{A}(=\nabla\times\mathbf{A})$ in terms of orthogonal generalized coordinates, where $\mathbf{A} = \sum_i A_i\mathbf{e}_i$.

Solution

As $\mathbf{A} = \sum_i A_i \mathbf{e}_i$, using the results from the previous Example 160, we obtain

$$\begin{aligned}
\nabla\times\mathbf{A} =& \nabla\times(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3) = \\
=& \nabla\times(A_1\mathbf{e}_1) + \nabla\times(A_2\mathbf{e}_2) + \nabla\times(A_3\mathbf{e}_3) = \\
=& \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u^3}(A_1h_1) - \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u^2}(A_1h_1) + \\
&+ \frac{\mathbf{e}_3}{h_1h_2}\frac{\partial}{\partial u^1}(A_2h_2) - \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u^3}(A_2h_2) + \\
&+ \frac{\mathbf{e}_1}{h_2h_3}\frac{\partial}{\partial u^2}(A_3h_3) - \frac{\mathbf{e}_2}{h_3h_1}\frac{\partial}{\partial u^1}(A_3h_3) = \\
=& \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u^2}(A_3h_3) - \frac{\partial}{\partial u^3}(A_2h_2)\right] + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u^3}(A_1h_1) - \frac{\partial}{\partial u^1}(A_3h_3)\right] + \\
&+ \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u^1}(A_2h_2) - \frac{\partial}{\partial u^2}(A_1h_1)\right] = \\
=& \frac{h_1\mathbf{e}_1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^2}(A_3h_3) - \frac{\partial}{\partial u^3}(A_2h_2)\right] + \frac{h_2\mathbf{e}_2}{h_3h_1h_2}\left[\frac{\partial}{\partial u^3}(A_1h_1) - \frac{\partial}{\partial u^1}(A_3h_3)\right] + \\
&+ \frac{h_3\mathbf{e}_3}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}(A_2h_2) - \frac{\partial}{\partial u^2}(A_1h_1)\right],
\end{aligned}$$

and thus rot$\mathbf{A}$ can be written in the following form

$$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u^1} & \dfrac{\partial}{\partial u^2} & \dfrac{\partial}{\partial u^3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}.$$

Exercise 162

Calculate $\nabla^2\psi$ in orthogonal generalized coordinates, where ψ is a scalar function.

Solution

According to Example 157 on p. 203 we have

$$\nabla\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial u^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial u^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial u^3}.$$

Let $\mathbf{A} = \nabla\psi$, then $A_1 = \dfrac{1}{h_1}\dfrac{\partial\psi}{\partial u^1}$, $A_2 = \dfrac{1}{h_2}\dfrac{\partial\psi}{\partial u^2}$, $A_3 = \dfrac{1}{h_3}\dfrac{\partial\psi}{\partial u^3}$, and this example comes down to Example 160a on p. 206.

$$\begin{aligned}
\nabla\cdot\mathbf{A} = \nabla\cdot\nabla\psi = \nabla^2\psi& \\
= \frac{1}{h_1h_2h_3}&\left[\frac{\partial}{\partial u^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u^2}\right) + \right. \\
&\left. + \frac{\partial}{\partial u^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u^3}\right)\right].
\end{aligned}$$

Exercise 163

Calculate the following in cylindrical cooridnates
a) $\nabla\Phi$, b) $\nabla\cdot\mathbf{A}$, c) $\nabla\times\mathbf{A}$, d) $\nabla^2\Phi$.

Solution

For cylindrical coordinates (ρ,ϕ,z)

$$u^1=\rho,\qquad u^2=\phi,\qquad u^3=z;$$
$$\mathbf{e}_1=\mathbf{e}_\rho,\qquad \mathbf{e}_2=\mathbf{e}_\phi,\qquad \mathbf{e}_3=\mathbf{e}_z;$$
$$h_1=h_\rho=1,\qquad h_2=h_\phi=\rho,\qquad h_3=h_z=1,$$

and thus $\mathbf{A}=A_\rho\mathbf{e}_1+A_\phi\mathbf{e}_2+A_z\mathbf{e}_3$.

a)

$$\begin{aligned}\nabla\Phi=&\frac{1}{h_1}\frac{\partial\Phi}{\partial u^1}\mathbf{e}_1+\frac{1}{h_2}\frac{\partial\Phi}{\partial u^2}\mathbf{e}_2+\frac{1}{h_3}\frac{\partial\Phi}{\partial u^3}\mathbf{e}_3=\\ =&\frac{1}{1}\frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho+\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi+\frac{1}{1}\frac{\partial\Phi}{\partial z}\mathbf{e}_z=\\ =&\frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho+\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi+\frac{\partial\Phi}{\partial z}\mathbf{e}_z.\end{aligned}$$

b)

$$\begin{aligned}\nabla\cdot\mathbf{A}=&\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}(A_1h_2h_3)+\frac{\partial}{\partial u^2}(A_2h_3h_1)+\frac{\partial}{\partial u^3}(A_3h_1h_2)\right]=\\ =&\frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial\rho}((\rho)(1)A_\rho)+\frac{\partial}{\partial\phi}((1)(1)A_\phi)+\frac{\partial}{\partial z}((1)(\rho)A_z)\right]=\\ =&\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{\partial A_\phi}{\partial\phi}+\frac{\partial}{\partial z}(\rho A_z)\right].\end{aligned}$$

c)

$$\nabla\times\mathbf{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial u^1} & \dfrac{\partial}{\partial u^2} & \dfrac{\partial}{\partial u^3}\\ A_1h_1 & A_2h_2 & A_3h_3\end{vmatrix}=\frac{1}{\rho}\begin{vmatrix}\mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}=$$
$$=\frac{1}{\rho}\left[\left(\frac{\partial A_z}{\partial\phi}-\frac{\partial}{\partial z}(\rho A_\phi)\right)\mathbf{e}_\rho+\left(\rho\frac{\partial A_\rho}{\partial z}-\rho\frac{\partial A_z}{\partial\rho}\right)\mathbf{e}_\phi+\left(\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right)\mathbf{e}_z\right].$$

Exercise 164

Write the Laplace equation in terms of parabolic coordinates.

Solution

For parabolic coordinates (ρ, ϕ, z)

$$u^1 = u, \qquad u^2 = v, \qquad u^3 = z;$$
$$\mathbf{e}_1 = \mathbf{e}_u, \qquad \mathbf{e}_2 = \mathbf{e}_v, \qquad \mathbf{e}_3 = \mathbf{e}_z;$$
$$h_1 = h_u = \sqrt{u^2+v^2}, \qquad h_2 = h_v = \sqrt{u^2+v^2}, \qquad h_3 = h_z = 1.$$

Then

$$\begin{aligned}\nabla^2\psi =& \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial \psi}{\partial v}\right) + \frac{\partial}{\partial z}\left((u^2+v^2)\frac{\partial \psi}{\partial z}\right)\right] = \\ =& \frac{1}{u^2+v^2}\left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right) + \frac{\partial^2\psi}{\partial z^2}.\end{aligned}$$

and the Laplace equation is $\nabla^2\psi = 0$, that is

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + (u^2+v^2)\frac{\partial^2\psi}{\partial z^2} = 0.$$

4.6.10 Surfaces in terms of orthogonal generalized coordinates

Exercise 165

Show that the square of the arc element of the curve $\mathbf{r} = \mathbf{r}(u, v)$ that lies in a plane, can be written in the following form

$$\mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2.$$

Solution

Given that

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v,$$

it follows that

$$\begin{aligned}\mathrm{d}s^2 =& \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = \\ =& \frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u^2 + 2\frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial v}\mathrm{d}u\,\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v^2 = \\ =& E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2,\end{aligned}$$

which yields

$$E = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial u} \qquad F = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial v} \qquad G = \frac{\partial \mathbf{r}}{\partial v}\frac{\partial \mathbf{r}}{\partial v}.$$

Exercise 166

Show that the element of the surface defined by the relation $\mathbf{r} = \mathbf{r}(u, v)$, can be expressed in the following form

$$\mathrm{d}S = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Solution

The surface element is given by

$$\mathrm{d}S = \left| \left(\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u\right) \times \left(\frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v\right) \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\,\mathrm{d}v =$$
$$= \sqrt{\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)}\,\mathrm{d}u\,\mathrm{d}v.$$

The value under the square root is

$$\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u}\right)\left(\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) - \left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}\right)\left(\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u}\right) = EG - F,$$

which represents the required result.

R Note. This idea can be demonstrated by observing the vector product, which yields

$$\mathbf{a} \times \mathbf{b} = ab \sin\alpha\,\mathbf{n} \quad \Rightarrow$$

$$|\mathbf{a} \times \mathbf{b}|^2 = a^2b^2\sin^2\alpha = a^2b^2(1 - \cos^2\alpha) =$$
$$= a^2b^2 - (ab\cos\alpha)^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2 =$$
$$= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix} \quad \Rightarrow$$
$$= |\mathbf{a} \times \mathbf{b}| = \sqrt{a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2}.$$

4.6.11 Generalized systems

Exercise 167

Let $\mathbf{A}$ be a vector and (u^1, u^2, u^3) and $(\overline{u}^1, \overline{u}^2, \overline{u}^3)$ two orthogonal curvilinear coordinate systems. Find the relation between contravariant components of this vector in the two coordinate systems.

Solution

Let us assume that the coordinate transformations between the Cartesian coordinate system and the (u^1, u^2, u^3) system, and the Cartesian coordinate system and the

$(\overline{u}^1, \overline{u}^2, \overline{u}^3)$ system, are given by

$$\begin{cases} x = x_1(u^1, u^2, u^3), & y = y_1(u^1, u^2, u^3), & z = z_1(u^1, u^2, u^3) \\ x = x_2(\overline{u}^1, \overline{u}^2, \overline{u}^3), & y = y_2(\overline{u}^1, \overline{u}^2, \overline{u}^3), & z = z_2(\overline{u}^1, \overline{u}^2, \overline{u}^3). \end{cases} \tag{4.227}$$

Then there exists a direct transformation from system (u^1, u^2, u^3) into system $(\overline{u}^1, \overline{u}^2, \overline{u}^3)$ defined by

$$u^1 = u^1(\overline{u}^1, \overline{u}^2, \overline{u}^3), \qquad u^2 = u^2(\overline{u}^1, \overline{u}^2, \overline{u}^3), \qquad u^3 = u^3(\overline{u}^1, \overline{u}^2, \overline{u}^3), \tag{4.228}$$

and vice versa. Based on the first set of equations, we obtain

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^1}\mathrm{d}u^1 + \frac{\partial \mathbf{r}}{\partial u^2}\mathrm{d}u^2 + \frac{\partial \mathbf{r}}{\partial u^3}\mathrm{d}u^3 = \alpha_1 \mathrm{d}u^1 + \alpha_2 \mathrm{d}u^2 + \alpha_3 \mathrm{d}u^3,$$

or

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \overline{u}^1}\mathrm{d}\overline{u}^1 + \frac{\partial \mathbf{r}}{\partial \overline{u}^2}\mathrm{d}\overline{u}^2 + \frac{\partial \mathbf{r}}{\partial \overline{u}^3}\mathrm{d}\overline{u}^3 = \overline{\alpha}_1 \mathrm{d}\overline{u}^1 + \overline{\alpha}_2 \mathrm{d}\overline{u}^2 + \overline{\alpha}_3 \mathrm{d}\overline{u}^3.$$

By equalizing the right sides we obtain

$$\alpha_1 \mathrm{d}u^1 + \alpha_2 \mathrm{d}u^2 + \alpha_3 \mathrm{d}u^3 = \overline{\alpha}_1 \mathrm{d}\overline{u}^1 + \overline{\alpha}_2 \mathrm{d}\overline{u}^2 + \overline{\alpha}_3 \mathrm{d}\overline{u}^3. \tag{4.229}$$

From (4.228)it follows that

$$\begin{aligned}
\mathrm{d}u^1 &= \frac{\partial u^1}{\partial \overline{u}^1}\mathrm{d}\overline{u}^1 + \frac{\partial u^1}{\partial \overline{u}^2}\mathrm{d}\overline{u}^2 + \frac{\partial u^1}{\partial \overline{u}^3}\mathrm{d}\overline{u}^3 \\
\mathrm{d}u^2 &= \frac{\partial u^2}{\partial \overline{u}^1}\mathrm{d}\overline{u}^1 + \frac{\partial u^2}{\partial \overline{u}^2}\mathrm{d}\overline{u}^2 + \frac{\partial u^2}{\partial \overline{u}^3}\mathrm{d}\overline{u}^3 \\
\mathrm{d}u^3 &= \frac{\partial u^3}{\partial \overline{u}^1}\mathrm{d}\overline{u}^1 + \frac{\partial u^3}{\partial \overline{u}^2}\mathrm{d}\overline{u}^2 + \frac{\partial u^3}{\partial \overline{u}^3}\mathrm{d}\overline{u}^3.
\end{aligned}$$

By substituting into (4.229) and equalizing the coefficients next to $\mathrm{d}\overline{u}^1$, $\mathrm{d}\overline{u}^2$ and $\mathrm{d}\overline{u}^3$, we obtain

$$\begin{cases}
\overline{\alpha}_1 = \alpha_1 \dfrac{\partial u^1}{\partial \overline{u}^1} + \alpha_2 \dfrac{\partial u^2}{\partial \overline{u}^1} + \alpha_3 \dfrac{\partial u^3}{\partial \overline{u}^1}, \\
\overline{\alpha}_2 = \alpha_1 \dfrac{\partial u^1}{\partial \overline{u}^2} + \alpha_2 \dfrac{\partial u^2}{\partial \overline{u}^2} + \alpha_3 \dfrac{\partial u^3}{\partial \overline{u}^2}, \\
\overline{\alpha}_3 = \alpha_1 \dfrac{\partial u^1}{\partial \overline{u}^3} + \alpha_2 \dfrac{\partial u^2}{\partial \overline{u}^3} + \alpha_3 \dfrac{\partial u^3}{\partial \overline{u}^3}.
\end{cases} \tag{4.230}$$

Now **A** can be expressed in the two systems

$$\mathbf{A} = C^1 \alpha_1 + C^2 \alpha_2 + C^3 \alpha_3 = \overline{C}^1 \overline{\alpha}_1 + \overline{C}^2 \overline{\alpha}_2 + \overline{C}^3 \overline{\alpha}_3, \tag{4.231}$$

where (C^1, C^2, C^3) and $(\overline{C}^1, \overline{C}^2, \overline{C}^3)$ are contravariant components of vector **A** in the two systems. Substituting the equation (4.230) into equation (4.231) we obtain

$$\begin{aligned}
&C^1 \alpha_1 + C^2 \alpha_2 + C^3 \alpha_3 = \overline{C}^1 \overline{\alpha}_1 + \overline{C}^2 \overline{\alpha}_2 + \overline{C}^3 \overline{\alpha}_3 = \\
&= \left(\overline{C}^1 \frac{\partial u^1}{\partial \overline{u}^1} + \overline{C}^2 \frac{\partial u^1}{\partial \overline{u}^2} + \overline{C}^3 \frac{\partial u^1}{\partial \overline{u}^3}\right) \alpha_1 + \left(\overline{C}^1 \frac{\partial u^2}{\partial \overline{u}^1} + \overline{C}^2 \frac{\partial u^2}{\partial \overline{u}^2} + \overline{C}^3 \frac{\partial u^2}{\partial \overline{u}^3}\right) \alpha_2 + \\
&\quad + \left(\overline{C}^1 \frac{\partial u^3}{\partial \overline{u}^1} + \overline{C}^2 \frac{\partial u^3}{\partial \overline{u}^2} + \overline{C}^3 \frac{\partial u^3}{\partial \overline{u}^3}\right) \alpha_3,
\end{aligned}$$

or

$$\begin{cases} C^1 = \overline{C}^1 \dfrac{\partial u^1}{\partial \overline{u}^1} + \overline{C}^2 \dfrac{\partial u^1}{\partial \overline{u}^2} + \overline{C}^3 \dfrac{\partial u^1}{\partial \overline{u}^3}, \\ C^2 = \overline{C}^1 \dfrac{\partial u^2}{\partial \overline{u}^1} + \overline{C}^2 \dfrac{\partial u^2}{\partial \overline{u}^2} + \overline{C}^3 \dfrac{\partial u^2}{\partial \overline{u}^3}, \\ C^3 = \overline{C}^1 \dfrac{\partial u^3}{\partial \overline{u}^1} + \overline{C}^2 \dfrac{\partial u^3}{\partial \overline{u}^2} + \overline{C}^3 \dfrac{\partial u^3}{\partial \overline{u}^3}, \end{cases} \tag{4.232}$$

or shortly

$$C^p = \overline{C}^1 \frac{\partial u^p}{\partial \overline{u}^1} + \overline{C}^2 \frac{\partial u^p}{\partial \overline{u}^2} + \overline{C}^3 \frac{\partial u^p}{\partial \overline{u}^3}, \qquad p = 1,2,3, \tag{4.233}$$

or

$$C^p = \sum_{q=1}^{3} \overline{C}^q \frac{\partial u^p}{\partial \overline{u}^q}, \qquad p = 1,2,3. \tag{4.234}$$

In the same way we obtain

$$\overline{C}^p = \sum_{q=1}^{3} C^q \frac{\partial \overline{u}^p}{\partial u^q}, \qquad p = 1,2,3. \tag{4.235}$$

Definition

If the coordinate transformation $\overline{u}^i = \overline{u}^i(u^j)$ transforms the system C^i according to the law (4.235), then this system defines a **contravariant tensor of the first order**.

From a geometrical point of view C^i determines contravariant coordinates of vector $\mathbf{C}$ in terms of the base of the coordinate system u^i, namely

$$\mathbf{C} = C^i \mathbf{g}_i.$$

This us why C^i is often called **contravariant vector** in literature.

Exercise 168

Determine how a covariant tensor (covariant vector coordinates) is transformed within a coordinate transformation.

Solution

Let us write the covariant components of vector $\mathbf{A}$ in systems (u^1, u^2, u^3) and $(\overline{u}^1, \overline{u}^2, \overline{u}^3)$

$$\mathbf{A} = c_1 \nabla u^1 + c_2 \nabla u^2 + c_3 \nabla u^3 = \overline{c}_1 \nabla \overline{u}^1 + \overline{c}_2 \nabla \overline{u}^2 + \overline{c}_3 \nabla \overline{u}^3. \tag{4.236}$$

Given that $\overline{u}^p = \overline{u}^p(u^1, u^2, u^3)$, where $p = 1,2,3$, it follows that

$$\begin{cases} \dfrac{\partial \overline{u}^p}{\partial x} = \dfrac{\partial \overline{u}^p}{\partial u^1} \dfrac{\partial u^1}{\partial x} + \dfrac{\partial \overline{u}^p}{\partial u^2} \dfrac{\partial u^2}{\partial x} + \dfrac{\partial \overline{u}^p}{\partial u^3} \dfrac{\partial u^3}{\partial x}, \\ \dfrac{\partial \overline{u}^p}{\partial y} = \dfrac{\partial \overline{u}^p}{\partial u^1} \dfrac{\partial u^1}{\partial y} + \dfrac{\partial \overline{u}^p}{\partial u^2} \dfrac{\partial u^2}{\partial y} + \dfrac{\partial \overline{u}^p}{\partial u^3} \dfrac{\partial u^3}{\partial y}, \\ \dfrac{\partial \overline{u}^p}{\partial z} = \dfrac{\partial \overline{u}^p}{\partial u^1} \dfrac{\partial u^1}{\partial z} + \dfrac{\partial \overline{u}^p}{\partial u^2} \dfrac{\partial u^2}{\partial z} + \dfrac{\partial \overline{u}^p}{\partial u^3} \dfrac{\partial u^3}{\partial z}. \end{cases} \tag{4.237}$$

Also

$$c_1 \nabla u^1 + c_2 \nabla u^2 + c_3 \nabla u^3 = \left(c_1 \frac{\partial u^1}{\partial x} + c_2 \frac{\partial u^2}{\partial x} + c_3 \frac{\partial u^3}{\partial x} \right) \mathbf{i} + \\ + \left(c_1 \frac{\partial u^1}{\partial y} + c_2 \frac{\partial u^2}{\partial y} + c_3 \frac{\partial u^3}{\partial y} \right) \mathbf{j} + \left(c_1 \frac{\partial u^1}{\partial z} + c_2 \frac{\partial u^2}{\partial z} + c_3 \frac{\partial u^3}{\partial z} \right) \mathbf{k}, \tag{4.238}$$

or

$$\bar{c}_1 \nabla \bar{u}^1 + \bar{c}_2 \nabla \bar{u}^2 + \bar{c}_3 \nabla \bar{u}^3 = \left(\bar{c}_1 \frac{\partial \bar{u}^1}{\partial x} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial x} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial x} \right) \mathbf{i} + \\ + \left(\bar{c}_1 \frac{\partial \bar{u}^1}{\partial y} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial y} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial y} \right) \mathbf{j} + \left(\bar{c}_1 \frac{\partial \bar{u}^1}{\partial z} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial z} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial z} \right) \mathbf{k}. \tag{4.239}$$

Equalizing coefficients next to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we obtain

$$\begin{cases} c_1 \frac{\partial u^1}{\partial x} + c_2 \frac{\partial u^2}{\partial x} + c_3 \frac{\partial u^3}{\partial x} = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial x} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial x} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial x}, \\ c_1 \frac{\partial u^1}{\partial y} + c_2 \frac{\partial u^2}{\partial y} + c_3 \frac{\partial u^3}{\partial y} = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial y} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial y} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial y}, \\ c_1 \frac{\partial u^1}{\partial z} + c_2 \frac{\partial u^2}{\partial z} + c_3 \frac{\partial u^3}{\partial z} = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial z} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial z} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial z}. \end{cases} \tag{4.240}$$

From equations (4.237) and (4.240), by equalizing the coefficients, we obtain

$$\begin{cases} c_1 = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial u^1} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial u^1} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial u^1}, \\ c_2 = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial u^2} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial u^2} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial u^2}, \\ c_3 = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial u^3} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial u^3} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial u^3}, \end{cases} \tag{4.241}$$

which can be written as

$$c_p = \bar{c}_1 \frac{\partial \bar{u}^1}{\partial u^p} + \bar{c}_2 \frac{\partial \bar{u}^2}{\partial u^p} + \bar{c}_3 \frac{\partial \bar{u}^3}{\partial u^p}, \tag{4.242}$$

or

$$c_p = \sum_{q=1}^{3} \bar{c}_q \frac{\partial \bar{u}^q}{\partial u^p} \qquad p = 1,2,3. \tag{4.243}$$

And analogously

$$\bar{c}_p = \sum_{q=1}^{3} c_q \frac{\partial u^q}{\partial \bar{u}^p} \qquad p = 1,2,3. \tag{4.244}$$

Definition

If the coordinate transformation $\bar{u}^i = \bar{u}^i(u^j)$ transforms the system c_i according to the law (4.244), then this system defines a **covariant tensor of the first order**.

Exercise 169

Let (u^1, u^2, u^3) be generalized coordinates.

a) Show that the following stands

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \left(\frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3} \right)^2,$$

where g_{pq} are metric coefficients next to $\mathrm{d}u^p \mathrm{d}u^q$ in the expressions for $\mathrm{d}s^2$ (see Example 156 on p. 203).

b) Show that the volume element in generalized orthogonal coordinates is equal to $\sqrt{g}\,\mathrm{d}u^1\,\mathrm{d}u^2\,\mathrm{d}u^3$.

Solution

a) Given that

$$g_{pq} = \alpha_p \cdot \alpha_q = \frac{\partial \mathbf{r}}{\partial u^p} \cdot \frac{\partial \mathbf{r}}{\partial u^q} = \frac{\partial x}{\partial u^p}\frac{\partial x}{\partial u^q} + \frac{\partial y}{\partial u^p}\frac{\partial y}{\partial u^q} + \frac{\partial z}{\partial u^p}\frac{\partial z}{\partial u^q} \qquad p, q = 1, 2, 3,$$

using the theorem on multiplication of determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_1A_1 + a_2A_2 + a_3A_3 & a_1B_1 + a_2B_2 + a_3B_3 & a_1C_1 + a_2C_2 + a_3C_3 \\ b_1A_1 + b_2A_2 + b_3A_3 & b_1B_1 + b_2B_2 + b_3B_3 & b_1C_1 + b_2C_2 + b_3C_3 \\ c_1A_1 + c_2A_2 + c_3A_3 & c_1B_1 + c_2B_2 + c_3B_3 & c_1C_1 + c_2C_2 + c_3C_3 \end{vmatrix}$$

we obtain

$$\left(\frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3} \right)^2 = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix}^2 =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}.$$

b) The volume element is given by the expression

$$\mathrm{d}V = \left| \left(\frac{\partial \mathbf{r}}{\partial u^1}\mathrm{d}u^1 \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u^2}\mathrm{d}u^2 \right) \times \left(\frac{\partial \mathbf{r}}{\partial u^3}\mathrm{d}u^3 \right) \right| = \left| \frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3} \right| \mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3$$
$$= \sqrt{g}\,\mathrm{d}u^1\,\mathrm{d}u^2\mathrm{d}u^3.$$

It should be noted that $\sqrt{g}$ is the absolute value of the Jacobian.

4.6.12 Various problems

Exercise 170

Express the following surfaces in terms of spherical coordinates

a) sphere $x^2+y^2+z^2=9$ c) paraboloid $z=x^2+y^2$
b) cone $z^2=3(x^2+y^2)$ d) surface $z=0$ e) surface $y=x$.

Solution

a) $r=3$, b) $\theta=\pi/6$, c) $r\sin^2\theta=\cos\theta$, d) $\theta=\pi/2$,
e) the plane $y=x$ consist of two half-planes $\phi=\pi/4$ and $\phi=5\pi/4$.

Exercise 171

If ρ,ϕ,z are cylindrical coordinates, draw each of the following examples and write their equations in Cartesian coordinates

a) $\rho=4, z=0$ b) $\rho=4$
c) $\phi=\pi/2$ d) $\phi=\pi/3, z=1$.

Solution

a) A circle in the z plane, $x^2+y^2=16, z=0$, b) Cylinder $x^2+y^2=16$, $z=z$.
c) The yz plane where $y\geq 0$, d) Straight line $y=\sqrt{3}x, z=1$ where $y\geq 0, x\geq 0$.

Exercise 172

For the spherical coordinate system, find

a) unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,
b) unit vectors of the Cartesian coordinate system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$.

Solution

a) $\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},$

$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},$

$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j},$

b) $\mathbf{i} = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi,$

$\mathbf{j} = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi,$

$\mathbf{k} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta.$

Exercise 173

Express the vector $\mathbf{A} = 2y\mathbf{i} - z\mathbf{j} + 3x\mathbf{k}$ in spherical coordinates and determine A_ρ, A_ϕ, A_z.

Solution

$$\begin{aligned}
\mathbf{A} &= A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi, \qquad \text{gde je}\\
A_r &= 2r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi,\\
A_\theta &= 2r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - 3r^2\sin^2\theta\cos\phi,\\
A_\phi &= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi.
\end{aligned}$$

Exercise 174

Prove that the following coordinate systems are orthogonal

a) parabolic cylindrical,
b) elliptic cylindrical, and
c) spheroid.

Exercise 175

Prove that a curvilinear coordinate system is orthogonal iff $g_{pq} = 0$ for $p \neq q$.

Exercise 176

Find the Jacobian $J = \dfrac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}$ for the following curvilinear systems

a) cylindrical,
b) spherical,
c) parabolic cylindrical,
d) elliptic cylindrical,
e) spheroid.

Solution

a) ρ, b) $r^2\sin\theta$, c) u^2+v^2,
d) $a^2(\operatorname{sh}^2 u+\sin^2 v)$, e) $a^3(\operatorname{sh}^2\xi+sin^2\eta)\operatorname{sh}\xi\sin\eta$

Exercise 177

Calculate $\int_V \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is a region bounded by the surfaces $z=x^2+y^2$ and $z=8-(x^2+y^2)$.

Solution

$\dfrac{256\pi}{15}$

Exercise 178

a) Describe the coordinate surfaces and coordinate lines for the system

$$x^2+y^2=2u_1\cos u_2,\quad xy=u_1\sin u_2,\quad z=u_3.$$

b) Show that the system is orthogonal.
c) Calculate the Jacobian of the transformation.
d) Show that u_1 and u_2 are related to cylindrical coordinates ρ and φ and find these relations.

Solution

c) $\dfrac{1}{2}$, d) $u_1=\frac{1}{2}\rho^2, u_2=2\phi$

Exercise 179

Find $\dfrac{\partial\mathbf{r}}{\partial u_1}$, $\dfrac{\partial\mathbf{r}}{\partial u_2}$, $\dfrac{\partial\mathbf{r}}{\partial u_3}$, ∇u_1, ∇u_2, and ∇u_3 in

a) cylindrical,
b) spherical and
c) parabolic cylindrical coordinates.

Show that $\mathbf{e}_1=\mathbf{E}_1$, $\mathbf{e}_2=\mathbf{E}_2$ and $\mathbf{e}_3=\mathbf{E}_3$ for these systems.

Solution

a) $$\frac{\partial \mathbf{r}}{\partial \rho} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}, \qquad \nabla\rho = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j},$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j}, \qquad \nabla\phi = \frac{-\sin\phi\mathbf{i}+\cos\phi\mathbf{j}}{\rho},$$

$$\frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}, \qquad \nabla z = \mathbf{k}$$

b) $$\frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\mathbf{i} + r\cos\theta\sin\phi\mathbf{j} - r\sin\theta\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\mathbf{i} + r\sin\theta\cos\phi\mathbf{j}$$

$$\nabla r = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

$$\nabla\theta = \frac{xz\mathbf{i}+yz\mathbf{j}-\sqrt{x^2+y^2}\mathbf{k}}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = \frac{\cos\theta\cos\phi\mathbf{i}+\cos\theta\sin\phi\mathbf{j}-\sin\theta\mathbf{k}}{r}$$

$$\nabla\phi = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2} = \frac{-\sin\phi+\cos\phi}{r\sin\theta}$$

c) $$\frac{\partial \mathbf{r}}{\partial u} = u\mathbf{i} + v\mathbf{j}, \qquad \nabla u = \frac{u\mathbf{i}+v\mathbf{j}}{u^2+v^2}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -v\mathbf{i} + u\mathbf{j}, \qquad \nabla v = \frac{-v\mathbf{i}+u\mathbf{j}}{u^2+v^2},$$

$$\frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}, \qquad \nabla z = \mathbf{k}.$$

Exercise 180

Express the equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \phi$ in terms of elliptic cylindrical coordinates.

Solution

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = a^2(\operatorname{sh}^2 u + \sin^2 v)\phi$$

Exercise 181

Express Schrodinger's equation (quantum mechanics)

$$\nabla^2\psi + \frac{8\pi^2 m}{h}(e - v(x,y,z)) = 0.$$

in terms of parabolic cylindrical coordinates, where m, h and E are constants.

Solution

$$\frac{1}{u^2+v^2}\left[\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right]+\frac{\partial^2\psi}{\partial z^2}+\frac{8\pi^2 m}{h^2}\left(E-W(u,v,z)\right)\psi=0,$$

where $W(u,v,z)=V(x,y,z)$.

Exercise 182

Express the equation $\frac{\partial U}{\partial t}=\kappa\nabla^2 U$ in terms of spherical coordinates if U is independent of

a) φ, b) φ and θ, c) r and t, d) φ, θ and t.

Solution

a) $\frac{\partial U}{\partial t}=\kappa\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)\right]$ b) $\frac{\partial U}{\partial t}=\kappa\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)\right]$

c) $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{\partial^2 U}{\partial\phi^2}=0$ d) $\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}U}{\mathrm{d}r}\right)=0.$

Exercise 183

Prove that in every coordinate system $\operatorname{div}\operatorname{rot}\mathbf{A}=0$ and $\operatorname{rot}\operatorname{grad}\phi=0$ is true.

Exercise 184

a) If $x=3u_1+u_2-u_3, y=u_1+2u_2+2u_3, z=2u_1-u_2-u_3$, find the volume of the cuboid bounded by $x=0, x=15$, $y=0, y=10$, $z=0, z=5$.
b) Find the relation between the volume and the Jacobian of the transformation.

Solution

a) 750,75; b) Jacobian=10.

Exercise 185

Let (x,y,z) and (u_1,u_2,u_3) be the coordinates of the same point in two systems.

a) Is the system orthogonal if $x = 3u_1 + u_2 - u_3$, $y = u_1 + 2u_2 + 2u_3$, $z = 2u_1 - u_2 - u_3$?
b) Find ds^2 and g for this system.

Solution

a) No. b) $ds^2 = 14du_1^2 + 6du_2^2 + 6du_1du_2 - 6du_1du_3 + 8du_2du_3$.

III

Solving differential equations

IV **Trigonometric Fourier series. Fourier integral**

5. Series Solutions of Differential Equations. Special functions

5.1 Functional series. Power series

Let $f_0(x), f_1(x), \cdots, f_k(x), \cdots$ be real functions defined for $\forall x \in [a,b] \subset \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Definition

The infinite sum of functions

$$f_0(x) + f_1(x) + \cdots + f_k(x) + \cdots = \sum_{k=0}^{\infty} f_k(x), \tag{5.1}$$

whose terms are functions $f_k(x)$ defined for $\forall x \in [a,b]$, is called a **functional series** (infinite functional series).

Definition

The **partial sum** of a functional series (5.1) has the following form

$$S_n(x) = \sum_{k=0}^{n} f_k(x), \quad (n - \text{a positive integer}). \tag{5.2}$$

Definition

The series (5.1) is **convergent**, for some $x = x_1 \in [a,b]$ if

$$\lim_{n\to\infty} S_n(x_1) = S(x_1) \neq \pm\infty. \tag{5.3}$$

If this limit value does not exists, then we say that the series is **divergent**.

If the series (5.1) is convergent for all values of the variable $x \in [a,b]$, then the sum of the series represents a function $f(x)$, for $x \in [a,b]$, and can be represented in the following form

$$f(x) = S_n(x) + R_n(x), \tag{5.4}$$

where S_n is the partial sum, and $R_n(x)$ the remainder. Then

$$f(x) = \lim_{n\to\infty} S_n(x), \quad \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - S_n(x)] = 0, \tag{5.5}$$

or

$$|f - S_n| = |R_n(x)| < \varepsilon$$

for each

$$n \geq N(\varepsilon, x) \quad \text{and for} \quad \forall x \in [a,b].$$

Definition

The series

$$\sum_{k=0}^{\infty} f_k(x)$$

is **absolutely convergent** for some $x = x_1 \in [a,b]$, if the series

$$\sum_{k=0}^{\infty} |\, f_k(x_1)\,|$$

is convergent.

Definition

The series (5.1) is **uniformly convergent** in the interval $[a,b]$, if for each arbitrary small $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$, which does not depend on x, such that

$$|R_n(x)| < \varepsilon \quad \text{for} \quad \forall n \geq N(\varepsilon) \quad \text{and} \quad \forall x \in [a,b].$$

R Note that a convergent series need not be uniformly convergent in the same interval.

Properties of uniformly convergent series

1 If the terms of a uniformly convergent (infinite) series are continuous functions of an independent variable $x \in [a,b]$, then their sum is also a continuous function of the variable x, in the same interval.

2 If the terms of a uniformly convergent series are continuous functions of an independent variable $x \in [a,b]$, then the integral of their sum is equal to the sum of their integrals, namely

$$\int_a^b f\,\mathrm{d}x = \int_a^b \left[\sum_{k=0}^{\infty} f_k\right] \mathrm{d}x = \sum_{k=0}^{\infty} \int_a^b f_k\,\mathrm{d}x, \quad \text{for } \forall x \in [a,b]. \tag{5.6}$$

3 Let the terms of a convergent series in the interval $[a,b]$ have continuous derivatives (in the same interval). If the series of derivatives is uniformly convergent for $x \in [a,b]$, then the initial series will also be uniformly convergent in the same interval, and can be differentiated term-by-term, namely

$$f' = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} f_k', \quad \text{for} \forall x \in [a,b]. \tag{5.7}$$

Power series

Definition

A series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots, \tag{5.8}$$

is called a **power series**. The constants a_0, a_1, ..., are called **series coefficients**.

It is assumed that the constants and the variable x belong to the set of real numbers (unless otherwise noted).

R Note that the power series is a functional series where $f_k(x) = a_k(x-x_0)^k$.

In the special case, when $x_0 = 0$, the power series has the following form

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots \tag{5.9}$$

Theorem 11 (Abel's theorem)

If the series (5.9) is convergent for $x = a$, it is absolutely convergent for each x when $|x| < |a|$.

Definition

For each power series (5.9) there exist a non-negative number R (including $+\infty$), such that the series is absolutely convergent for $\forall x \in (-R, R)$, that is, for $|x| < R$, and divergent for $\forall x$ outside this interval. The number R is called the **convergence radius**, and the interval $(-R, R)$ the **convergence interval**.

Operations on power series

1° Each power series, which is convergent for $x \in (-R,R)$, can be integrated in the interval $[0,x]$, where $|x| < R$, and in that case the integral of the sum is the sum of the integrals, namely

$$\int_0^x f(x)\,\mathrm{d}x = \int_0^x \left(\sum_{k=0}^{\infty} a_k x^k\right)\mathrm{d}x = \sum_{k=0}^{\infty}\left(\int_0^x a_k x^k\,\mathrm{d}x\right), \quad \text{for } |x| < R. \tag{5.10}$$

2° Each power series, which is convergent for $x \in (-R, R)$, can be differentiated in the interval $[0, x]$, where $|x| < R$, and in that case the derivative of the sum is the sum of the derivatives, namely

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}(a_k x^k)}{\mathrm{d}x}, \quad \text{for } |x| < R. \tag{5.11}$$

3° By adding (subtracting) two convergent power series, a convergent power series is obtained, whose convergence radius is not less than the smaller convergence radius of the two given series. Namely, let

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k x^k, \quad |x| < R, \\ g(x) &= \sum_{k=0}^{\infty} b_k x^k, \quad |x| < R', \quad R' \le R, \end{aligned} \tag{5.12}$$

be two power series, then their sum (difference) is the following power series

$$f(x) \pm g(x) = \sum_{k=0}^{\infty} (a_k \pm b_k) x^k, \tag{5.13}$$

which is convergent in the interval $(-R', R')$.

4° By multiplying two convergent power series, a convergent power series is obtained, whose convergence radius is not less than the smaller convergence radius of the two given series. Namely, let

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k x^k, \quad |x| < R, \\ g(x) &= \sum_{k=0}^{\infty} b_k x^k, \quad |x| < R', \quad R' \le R, \end{aligned} \tag{5.14}$$

be two power series, then their product is the following power series

$$f(x) \cdot g(x) = \sum_{k=0}^{\infty} (a_0 \cdot b_k + a_1 \cdot b_{k-1} + \cdots + a_k \cdot b_0) x^k, \tag{5.15}$$

which is convergent in the interval $(-R', R')$.

Theorem 12

If a power series has a positive convergence radius ($R > 0$), and its sum is equal to zero, then all terms of this series are equal to zero.

Definition

A function $f(x)$ is called **analytical** at point $x = x_0$, if it can be represented by a power series of $(x - x_0)$, with a convergence radius $R > 0$.

5.2 Series Solutions of Differential Equations

It is known from mathematical analysis that a homogeneous linear differential equation with constant coefficients can be solved by algebraic methods, and that the solutions are elementary functions. For example, consider a homogeneous second-order linear differential equation with constant coefficients:

$$ay'' + by' + cy = 0, \tag{5.16}$$

where a, b, c are constants, and $y = y(x)$. It is assumed that its solution has the form

$$y = C e^{\alpha x}, \tag{5.17}$$

where α and C are constants. The constant α is determined from the condition that the assumed solution satisfies the initial equation. This condition, by substituting (5.17) into (5.16), is reduced to an algebraic (quadratic) equation.

$$a\alpha^2 + b\alpha + c = 0, \tag{5.18}$$

which has two solutions (α_1, α_2). These solutions, in the general case, are complex numbers. The final solution of the initial equation (obtained by applying the superposition principle) is

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}. \tag{5.19}$$

In practice, cases when the coefficients of equation (5.16) are not constant, but rather depend on x, are more frequent. In addition, equations can also be inhomogeneous, which makes their solution even more difficult. The solutions of these differential equations are often functions that are not elementary. This chapter will outline some of them (the most commonly used).

5.2.1 Solutions of Differential Equations using Power Series

Power series are most often used to solve differential equations when the solution cannot be obtained in closed form. This method is natural and relatively simple. Namely, all functions that appear in the observed differential equation are developed into a power series of $x - x_0$ (see (5.8)), or, in the special case, of x (x_0=0). Then a solution in the form of a power series is assumed

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m,$$

and the corresponding derivatives are determined and substituted in the initial equation. Finally, by equating the coefficients next to the same powers of x, we obtain the unknown coefficients a_m, and consequently a solution (in the form of a series).

We will demonstrate this technique on the examples of Legendre[1] and Bessel[2] equations. However, let us first state a theorem of importance for solving these equations.

[1]Adrien-Marie Legendre (1752-1833), French mathematician. He made substantial contributions in the field of special functions, elliptic integrals, number theory and calculus of variations. His book *Éléments de géométrie* (1794) was very well known and had 12 editions in a period of less than 30 years.

[2]Friedrich Wilhelm Bessel (1784-1846), German astronomer and mathematician. His work on Bessel functions appeared in 1826.

Theorem 13

If the functions p, q and r, in differential equation

$$y'' + p(x)y' + q(x)y = r(x) \tag{5.20}$$

are analytical at point $x = x_0$, then each solution of the equation (5.20) is analytical at point $x = x_0$, and can be represented by a power series of $(x - x_0)$, with a convergence radius $R > 0$.

R Note. When applying this theorem, it is important to write the equation in the form (5.20), so that the coefficient next to the highest derivative is equal to 1.

Finally, let us note that this method is of significant practical importance due to the possibility of calculating numerical values for a series.

5.3 Legendre: equation, function, polynomial

Definition

Differential equation of the form

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0, \tag{5.21}$$

where k is a known real number, is known in the literature as **Legendre equation**. Solutions of the equation (5.21) are called **Legendre functions**.

This equation appears in numerous problems of mathematical physics, as well as in solving partial differential equations.

According to the previous note, the coefficient next to y'' should be equal to 1, and thus dividing by $(1 - x^2)$ we obtain

$$y'' - \frac{2x}{1 - x^2}y' + \frac{k(k+1)}{1 - x^2}y = 0. \tag{5.22}$$

As the conditions of Theorem 13 are fulfilled

$$\left(p = -\frac{2x}{1 - x^2}, q = \frac{k(k+1)}{1 - x^2}, r = 0\right),$$

that is, the corresponding coefficients (p, q, r) are analytical functions for $x = 0$, the solution can be represented by the power series

$$y = \sum_{i=0}^{\infty} a_i x^i. \tag{5.23}$$

Coefficients a_i are determined from the condition that this solution satisfies the initial equation for each x. By substituting the assumed solution into the initial equation, we obtain

$$(1 - x^2)\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - 2x\sum_{i=1}^{\infty} i a_i x^{i-1} + k(k+1)\sum_{i=0}^{\infty} a_i x^i = 0, \tag{5.24}$$

or

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2\sum_{i=1}^{\infty} i a_i x^i + k(k+1)\sum_{i=0}^{\infty} a_i x^i = 0. \tag{5.25}$$

This relation can be expanded as follows

$$\begin{aligned}
2\cdot 1\cdot a_2 + 3\cdot 2\cdot a_3\cdot x + 4\cdot 3\cdot a_4\cdot x^2 + \cdots + (s+2)(s+1)a_{s+2}\cdot x^s + \cdots \\
-2\cdot 1\cdot a_2\cdot x^2 - \cdots - s(s-1)a_s x^s - \cdots \\
-2\cdot 1\cdot a_1\cdot x - 2\cdot 2\cdot a_2\cdot x^2 - \cdots - 2\cdot s\cdot a_s\cdot x^s - \cdots \\
k(k+1)a_0 + k(k+1)a_1\cdot x + k(k+1)a_2\cdot x^2 + \cdots + k(k+1)\cdot a_s\cdot x^s + \cdots = 0.
\end{aligned}$$

From the condition that the above relation must be an identity, according to Theorem 12, we obtain

$$\begin{aligned}
2a_2 + k(k+1)a_0 = 0, &\quad \text{coefficient next to } x^0 \\
6a_3 + [-2 + k(k+1)]a_1 = 0, &\quad \text{coefficient next to } x^1 \\
(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + k(k+1)]a_s = 0, &\quad \text{coefficient next to } x^s.
\end{aligned}$$

From the last relation we obtain the so called recurrence formulas for determining the coefficients a_i

$$a_{s+2} = -\frac{(k-s)(k+s+1)}{(s+2)(s+1)}a_s, \quad s = 0,1,2,\ldots \tag{5.26}$$

It is visible from these relations that all coefficients are determined except a_0 and a_1, which remain arbitrary. Thus, all other coefficients ca be expressed in terms of these two. For example

$$\begin{aligned}
a_2 &= -\frac{k(k+1)}{2\cdot 1}a_0 = -\frac{k(k+1)}{2!}a_0 \\
a_3 &= -\frac{(k-1)(k+2)}{3\cdot 2}a_1 = -\frac{(k-1)(k+2)}{3!}a_1 \\
a_4 &= -\frac{(k-2)(k+3)}{4\cdot 3}a_2 = \frac{(k-2)k(k+1)(k+3)}{4!}a_0 \\
a_5 &= -\frac{(k-3)(k+4)}{5\cdot 4}a_3 = \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1
\end{aligned}$$

etc.

The above examples show that the even coefficients can be expressed in terms of a_0, and the odd ones in terms of a_1, and thus the initial solution can be expressed in the form

$$y = a_0 y_1(x) + a_1 y_2(x), \tag{5.27}$$

where

$$y_1(x) = 1 - \frac{k(k+1)}{2!}x^2 + \frac{(k-2)k(k+1)(k+3)}{4!}x^4 + \cdots$$

$$y_2(x) = x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-3)(k-1)(k+2)(k+4)}{5!}x^5 + \cdots$$

This series converges for $|x| < 1$.

R Note that y_1 contains only even powers, and y_2 only odd ones, and thus their quotient is not constant. It follows that y_1 and y_2 are not proportional, that is, these two functions are linearly independent. Thus, the function $y = a_0 y_1 + a_1 y_2$ represents the general solution of the initial (Legendre) equation in the interval $-1 < x < 1$.

Legendre polynomials

Observing the structure of coefficients a_{s+2} (5.26), where $s = 0, 1, 2, \ldots$, we can see that if k in the initial equation is an integer, then some of the coefficients a_{s+2} can be equal to zero. For example, for $k = s$ we have $a_{k+2} = a_{k+4} = \cdots = 0$. If k is an even number, then y_1 is reduced to a polynomial of power k. If k is an odd number, then y_2 is reduced to a polynomial of power k.

From (5.26) it follows that

$$a_s = -\frac{(s+1)(s+2)}{(k-s)(k+s+1)} a_{s+2}, \quad s \leq k-2. \tag{5.28}$$

From this relation, all coefficients different from zero can be determined in terms of a_k, namely the coefficient next to the highest power of x in the polynomial. This coefficient remains arbitrary. For convenience, it can be defined by the relation

$$a_k = \begin{cases} 1, & k = 0, \\ \dfrac{(2k)!}{2^k (k!)^2} = \dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!}, & k = 1, 2, \ldots \end{cases} \tag{5.29}$$

For a_k defined in this way, we obtain from (5.28)

$$\begin{aligned} a_{k-2} &= -\frac{k(k-1)}{2(2k-1)} a_k = -\frac{k(k-1)(2k)!}{2(2k-1)2^k (k!)^2} = \\ &= -\frac{(2k-2)!}{2^k (k-1)!(k-2)!} \\ &\vdots \qquad\qquad \vdots \\ a_{k-2m} &= (-1)^m \frac{(2k-2m)!}{2^k m!(k-m)!(k-2m)!}. \end{aligned}$$

Definition

The polynomial defined by the relation

$$P_k(x) = \sum_{m=0}^{M} (-1)^m \frac{(2k-2m)!}{2^k m!(k-m)!(k-2m)!} x^{k-2m} \tag{5.30}$$

is called **Legendre polynomial** of power k, where $M = k/2$ or $(k-1)/2$ is a whole number.

This polynomial represents the solution of Legendre differential equation (5.21). Let us write down some of these polynomials

$$\begin{aligned} P_0 &= 1; & P_1 &= x \\ P_2 &= \frac{1}{2}(3x^2 - 1); & P_3 &= \frac{1}{2}(5x^3 - 3x) \\ P_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3); & P_5 &= \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

Their graphs are represented in Figure 5.1.

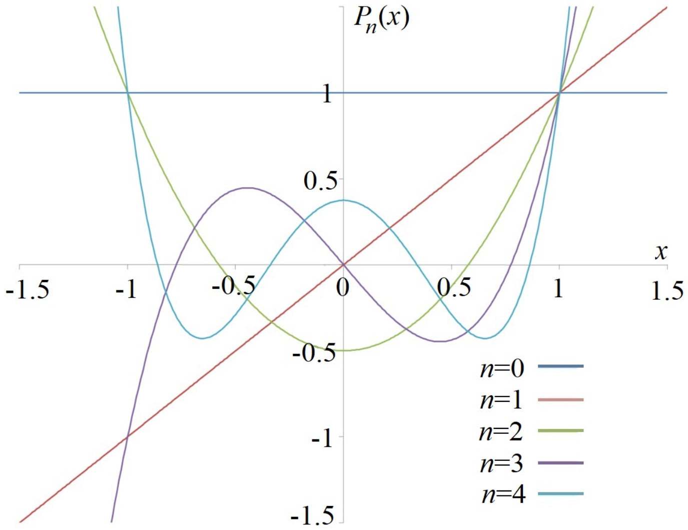

Figure 5.1: Legendre polynomials.

Starting from the binomial formula, applied to $(x^2-1)^n$, and differentiating this expression n times, term by term, we obtain the so called Rodrigues[3] formula

$$P_k(x) = \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\left(x^2-1\right)^k\right]. \tag{5.31}$$

An example of application of Legendre polynomials in geophysics is geomagnetic potential [72]. Namely, for calculating the magnetic potential at the Earth's surface, a function is used of the form

$$U = R\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[g_o^m\cos(m\lambda)+h_o^m\sin(m\lambda)\right]P_n^m\left[\cos\Theta\right],$$

where R is Earth's radius, g_o^m and h_o^m coefficients that depend on the basic characteristics of the magnetic field, and P_n^m are Legendre polynomials (see Example 197, p. 282).

5.4 Bessel equation. Bessel functions

Let us first introduce some concepts and theorems that are going to be used when solving Bessel equation.

Observe the linear differential equation of nth order

$$a_0(x)y^{(n)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0. \tag{5.32}$$

Definition

The point x_0, at which the condition $a_0(x_0)=0$ is fulfilled, is called a **singular point**.

[3] Olinde Rodrigues (1794-1851), French mathematician and economist.

Definition

The point x_0 is a **regular singular point**, of a differential equation if this equation can be represented around the point x_0 in the form

$$(x-x_0)^n y^{(n)} + b_1(x)(x-x_0)^{(n-1)} y^{(n-1)} + \cdots + b_n(x)y = 0, \tag{5.33}$$

where $b_i(x)$, $i = 1, 2, \ldots, n$, are analytical functions at point x_0.

As we will tackle later only second-order differential equations, let us consider the equation

$$L(y) \equiv (x-x_0)^2 y'' + b(x)(x-x_0)y' + c(x)y = 0. \tag{5.34}$$

Without losing generality, and for the sake of simpler writing, we will assume that $x_0 = 0$[4] , and thus obtain

$$L(y) \equiv x^2 y'' + b(x)xy' + c(x)y = 0. \tag{5.35}$$

We assume that $b(x)$ and $c(x)$ are analytical functions at point x, that is, that there exists a number $R > 0$ such that they can be represented by power series

$$b(x) = \sum_{k=0}^{\infty} b_k x^k, \quad c(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{5.36}$$

that converge in the interval $|x| < R$.

We will seek the solution in the form of a so called generalized power series

$$y(x) = x^{\lambda} \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0,\ x > 0. \tag{5.37}$$

Theorem 14 (Frobenius[5]method)

Every differential equation of the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0, \tag{5.38}$$

where functions $b(x)$ and $c(x)$ are analytical at point $x = 0$, has a solution in the form

$$y(x) = x^{\lambda} \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0, \tag{5.39}$$

where λ can be any number (real or complex). λ is chosen so that $a_0 \neq 0$.

Substituting the assumed solution into the initial equation, for $a_0 \neq 0$ and $k = 0$ we obtain the equation for determining λ

$$\lambda(\lambda-1) + b(0)\lambda + c(0) = 0. \tag{5.40}$$

[4]We could introduce a new variable $\bar{x} = x - x_0$ and thus formally obtain the same equation, as for $x_0 = 0$.

[5]Georg Frobenius (1849-1917), German mathematician, who contributed significantly to matrix theory and group theory.

This equation is known as the **index equation** of the differential equation (5.38). The remaining coefficients a_k can now be determined using λ and a_0 , and the formal solution of the initial equation can be written in the following form

$$y(x,\lambda) = a_0 x^\lambda + x^\lambda \sum_{k=1}^{\infty} a_k(\lambda) x^k. \tag{5.41}$$

However, the solution is formal, as the convergence of this series has not been proved. The proof can be found in various books from this area. See, for example, [62].

Theorem 15

Let us observe the differential equation in the following form

$$x^2 y'' + x b(x) y' + c(x) y = 0, \tag{5.42}$$

and assume that $b(x)$ and $c(x)$ are analytical functions at point $x = 0$. If these functions can be replaced by series that are convergent for $|x| < R$, and λ_i ($\mathrm{Re}\lambda_1 > \mathrm{Re}\lambda_2,\ i = 1,2$) are the solutions of the index equation

$$\lambda(\lambda - 1) + b(0)\lambda + c(0) = 0, \tag{5.43}$$

then

a) differential equation (5.42) has two linearly independent solutions

$$y_i(x) = |x|^{\lambda_i} \sum_{k=0}^{\infty} a_k^{(i)} x^k, \quad a_0^{(i)} = 1,\ i = 1,2, \tag{5.44}$$

if λ_1 and λ_2 are not equal, and their difference $\underline{\lambda_1 - \lambda_2 \text{ is not a positive integer}}$. The respective series (y_i) are convergent for $0 < |x| < R$.

b) Differential equation (5.42) has two solutions in the format

$$y_1(x) = |x|^{\lambda_1} \sum_{k=0}^{\infty} a_k^{(1)} x^k = |x|^{\lambda_1} r_1(x),$$

$$y_2(x) = |x|^{\lambda_1+1} \sum_{k=0}^{\infty} a_k^{(2)} x^k + c y_1(x) \cdot \log|x| = |x|^{\lambda_1+1} r_2(x) + c y_1(x) \cdot \log|x|,$$

if $\underline{\lambda_1 = \lambda_2}$. The respective power series $r_1(x)$ and $r_2(x)$ are convergent for $0 < |x| < R$ and $r_1(0) \neq 0$.

c) Differential equation (5.42) has two solutions in the format

$$y_1(x) = |x|^{\lambda_1} q_1(x),$$
$$y_2(x) = |x|^{\lambda_2} q_2(x) + c y_1(x) \cdot \log|x| = |x|^{\lambda_1} r_2(x) + c y_1(x) \cdot \log|x|,$$

if the difference $\underline{\lambda_1 - \lambda_2 \text{ is a positive integer}}$. The power series $q_1(x)$ and $q_2(x)$ are convergent for $0 < |x| < R$ and $q_i(0) \neq 0$. c is a constant, which may also be equal to zero.

5.4.1 Bessel equation

One of the more important equations in mathematical applications is the so called **Bessel** or **cylindrical equation**

$$L(y) \equiv x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \tag{5.45}$$

where ν is a constant ($\mathrm{Re}\,\nu > 0$).

This equation occurs in problems of oscillations, electrostatic fields, heat conduction, etc. We will further assume that ν is a real parameter.

Let us demonstrate the procedure described above on this equation.

First, we can observe a regular singularity at point $x = 0$. In this case, the index equation

$$\lambda(\lambda - 1) + 1 \cdot \lambda + (0 - \nu^2) = 0$$

is a quadratic equation

$$\lambda^2 - \nu^2 = 0, \tag{5.46}$$

with the following solutions

$$\lambda_1 = \nu, \quad \lambda_2 = -\nu.$$

Let us assume that $\nu = 0$, from where it follows that $\lambda_1 = \lambda_2 = 0$. According to Theorem 15 b), p.233, in this case we obtain the following solutions

$$\begin{aligned} y_1(x) &= |x|^0 r_1(x) \\ y_2(x) &= |x|^{0+1} r_2(x) + y_1(x) \log|x|, \end{aligned}$$

or if we observe the case where $x > 0$

$$\begin{aligned} y_1(x) &= r_1(x) \\ y_2(x) &= x r_2(x) + y_1(x) \log x. \end{aligned}$$

According to the initial assumptions, $r_i(x)$ are analytical functions for $x = 0$, and each of them can thus be represented by a series that converges for all finite values of x. Let us first determine r_1

$$r_1(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0 \tag{5.47}$$

and calculate $L(r_1)$. Given that

$$r_1'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{i} \quad r_1''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}, \tag{5.48}$$

it follows that

$$L(r_1) \equiv x^2 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} + x^2 \sum_{k=0}^{\infty} a_k x^k = 0, \tag{5.49}$$

being one of the solutions (y_1) of the initial equation.

From the previous relation we obtain

$$
\begin{aligned}
& a_1 x + a_0 x^2 + a_1 x^3 + \sum_{k=2}^{\infty}\left[k(k-1)a_k x^k + k a_k x^k + a_k x^{k+2}\right] = \\
&= a_1 x + \sum_{k=2}^{\infty}\left[k(k-1)a_k x^k + k a_k x^k\right] + a_0 x^2 + a_1 x^3 + \sum_{k=2}^{\infty} a_k x^{k+2} = \\
&= a_1 x + \sum_{k=2}^{\infty}\left[k(k-1)a_k x^k + k a_k x^k + a_{k-2} x^k\right] = \\
&= a_1 x + \sum_{k=2}^{\infty}\left\{\left[k(k-1)+k\right]a_k + a_{k-2}\right\} x^k = 0.
\end{aligned} \tag{5.50}
$$

In order for this relation to be identically satisfied, it is necessary that the coefficients next to all powers be equal to zero. From this condition it follows that

$$
a_1 = 0, \quad \left(k^2 - k + k\right) a_k + a_{k-2} = 0,
$$

or

$$
a_1 = 0, \quad a_k = -\frac{a_{k-2}}{k^2}, \quad \text{za } k = 2,3,\ldots \tag{5.51}
$$

From the last relation we can observe that all even coefficients ($k = 2s+2$, $s = 0,1,2,\ldots$) are expressed in terms of a_0,whereas the odd ones ($k = 2s+1$) are expressed in terms of a_1. As we have obtained that $a_1 = 0$, it follows that all odd coefficients are equal to zero.

For even coefficients we obtain

$$
a_0 \neq 0,\ a_2 = -\frac{a_0}{4},\ a_4 = -\frac{a_2}{16} = -\frac{1}{16}\left(-\frac{a_0}{4}\right) = \frac{1}{4 \cdot 16} \cdot a_0, \tag{5.52}
$$

etc.

By continuing this procedure we obtain the following relations

$$
\begin{aligned}
a_{2s+1} &= 0, \\
a_{2s} &= \left[\frac{(-1)^s}{(s!)^2 \cdot 2^{2s}}\right] a_0.
\end{aligned} \tag{5.53}
$$

If we assume that $a_0 = 1$, the series r_1 has the form

$$
r_1(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s!)^2 2^{2s}} x^{2s}, \tag{5.54}
$$

which converges for each finite x.

Definition

The function defined by the relation

$$
J_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s!)^2}\left(\frac{x}{2}\right)^{2s} \tag{5.55}
$$

is called **Bessel function of the first kind of zero order**.

Thus, $r_1 = J_0$ represents the first particular solution of differential equation (5.45).

We will search for the second particular solution, according to Theorem 15, p.233, in the form[6]

$$y_2(x) = \sum_{k=0}^{\infty} b_k x^k + J_0 \log x, \quad \text{where } b_0 = 0. \tag{5.56}$$

As in the case of the first particular solution, let us first find

$$\begin{aligned} L(y_2) &= x^2 y_2'' + x y_2' + x^2 y_2 = \\ &= b_1 x + 2^2 b_2 x^2 + \sum_{k=3}^{\infty} \left(k^2 b_k + b_{k-2}\right) x^k + 2x J_0' + L(J_0) \log x. \end{aligned} \tag{5.57}$$

As y_2 and J_0 are solutions of the initial differential equation ($L(y_2) = 0$, $L(J_0) = 0$), we obtain

$$2 \sum_{s=1}^{\infty} \frac{(-1)^s 2s}{(s!)^2} \left(\frac{x}{2}\right)^{2s} + \sum_{k=3}^{\infty} \left(k^2 b_k + b_{k-2}\right) x^k + 2^2 b_2 x^2 + b_1 x = 0. \tag{5.58}$$

From here coefficients b_i can be determined. For odd coefficients we obtain

$$(2s+1)^2 b_{2s+1} = -b_{2s-1}, \quad s = 1, 2, \ldots,$$

and as $b_1 = 0$, the odd coefficients are

$$b_{2s+1} = 0, \quad s = 0, 1, 2, \ldots \tag{5.59}$$

For even coefficients we obtain

$$(2s)^2 b_{2s} + b_{2s-2} = \frac{(-1)^{s+1} s}{2^{2s-2} (s!)^2}, \quad s = 2, 3, \ldots \tag{5.60}$$

It can be shown that the last relation yields

$$b_{2s} = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}\right) \frac{(-1)^{s-1}}{2^{2s} (s!)^2}, \quad s = 1, 2, \ldots \tag{5.61}$$

The second particular solution of the equation (5.45) is thus also determined as

$$y_2 = J_0 \log x - \sum_{s=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{s}\right) \frac{(-1)^s}{(s!)^2} \left(\frac{x}{2}\right)^{2s}. \tag{5.62}$$

[6] Note that for the first part of the solution we had

$$xr_2 = x \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+1}, \; k+1 = s \quad \Rightarrow$$

$$xr_2 = \sum_{s=1}^{\infty} c_{s-1} x^s, \quad \text{and by substituting } c_{s-1} = b_s \quad \text{we obtain}$$

$$xr_2 = \sum_{s=1}^{\infty} b_s x^s = \sum_{s=0}^{\infty} b_s x^s, \text{ for } s = 0, b_0 = 0.$$

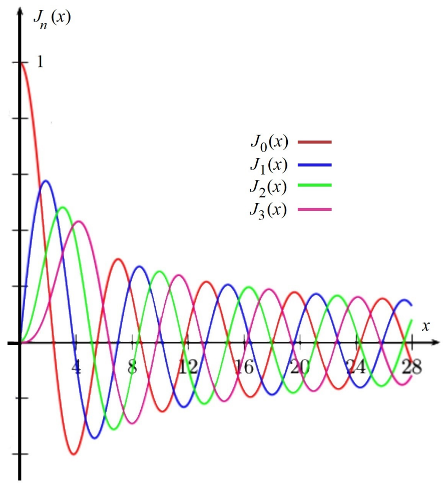

Figure 5.2: Bessel functions of the first kind.

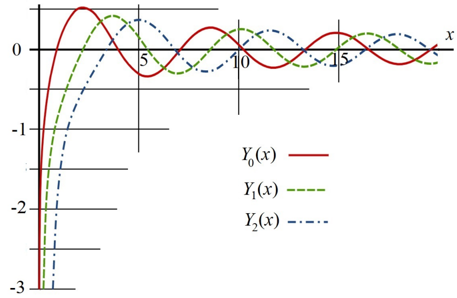

Figure 5.3: Bessel functions of the second kind.

Definition

The function defined by the relation

$$K_0 = J_0 \log x - \sum_{s=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{s}\right) \frac{(-1)^s}{(s!)^2} \left(\frac{x}{2}\right)^{2s} \tag{5.63}$$

is called **Bessel function of the second kind of zero order**.

This function i known in literature also as Neumann[7] or Macdonald function of zero order.

Let us finally summarize. For $\nu = 0$, Bessel equation (5.45) has two linearly independent solutions

$$y_1 = J_0, \quad y_2 = K_0, \tag{5.64}$$

that is, in this case the solution of the equation is

$$y = aJ_0 + bK_0, \tag{5.65}$$

where a and b are arbitrary constants.

When solving the index equation that corresponds to Bessel equation (5.45) we have assumed that $\nu = 0$. Now consider the case when $\nu \neq 0$. In this case, $\lambda_1 \neq \lambda_2$ and

$$\lambda_1 - \lambda_2 = 2\nu. \tag{5.66}$$

As we have assumed that ν is a real parameter, we will observe two cases, when the difference $\lambda_1 \neq \lambda_2$ is an integer, and when it is not an integer.

Consider the case when 2ν is not integer. According to Theorem 15 a), in this case there are two linearly independent solutions of the form

$$y_i = |x|^{\lambda_i} \sum_{k=0}^{\infty} a_k^{(i)} x^k, \quad i = 1, 2. \tag{5.67}$$

[7]Carl Neumann (1832-1925), German mathematician and physicist.

Lest us first find y_1, assuming that $x > 0$, for $\lambda_1 = \nu$

$$y_1 = x^\nu \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0, \tag{5.68}$$

where this series is convergent for each finite x.

As in the previous case, from the condition

$$L(y_1) = (\nu+1)a_1 x^{\nu+1} + x^\nu \sum_{k=2}^{\infty} \left[(k+\nu)^2 a_k + a_{k-2}\right] x^k = 0, \tag{5.69}$$

we can obtain

$$\begin{aligned} a_{2s+1} &= 0, \quad s = 0, 1, 2, \ldots \\ a_{2s} &= \frac{(-1)^s a_0}{2^{2s} s!(\nu+1)(\nu+2)\cdots(\nu+s)}. \end{aligned} \tag{5.70}$$

The particular solution has the form

$$y_1(x) = a_0 x^\nu + a_0 x^\nu \sum_{s=1}^{\infty} \frac{(-1)^s}{s!(\nu+1)(\nu+2)\cdots(\nu+s)} \left(\frac{x}{2}\right)^{2s}. \tag{5.71}$$

It is convenient to choose the following value for a_0 (in order to connect the above relation to another special function)

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}, \tag{5.72}$$

where Γ is the so called gamma function.[8] For a_0 chosen in this way y_1 becomes

$$y_1 = \left(\frac{x}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s}. \tag{5.73}$$

Definition

The function $J_\nu(x)$, defined by the relation

$$J_\nu = \left(\frac{x}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s}. \tag{5.74}$$

is called the **Bessel function of the first kind of order ν**.

As $\lambda_1 - \lambda_2$ is not an integer, we will search for the second particular solution in the following form

$$y_2 = x^{\lambda_2} \sum_{k=0}^{\infty} b_k x^k = x^{-\nu} \sum_{k=0}^{\infty} b_k x^k. \tag{5.75}$$

[8]The Γ function will be discussed later in more detail. At this point we shall only state its definition and some of its properties, to make the text easier to follow.

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} \mathrm{d}x, \text{ for } \nu > 0.$$

For the function defined in this way the following stands

$$\Gamma(\nu+1) = \nu\Gamma(\nu), \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}.$$

In this case, using the same procedure, we obtain

$$y_2 = J_{-\nu}(x), \tag{5.76}$$

where $J_{-\nu}$ is a Bessel function, defined by the expression

$$J_{-\nu} = \left(\frac{x}{2}\right)^{-\nu} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s-\nu+1)} \left(\frac{x}{2}\right)^{2s}. \tag{5.77}$$

There is one case remaining, namely when ν is a positive integer, that is, $\nu = n$, where n denotes a natural number. In this case, as the first particular solution we obtain (see Theorem 15, p.233)

$$y_1 = J_n(x), \tag{5.78}$$

and as the second particular solution

$$y_2 = x^n \sum_{k=0}^{\infty} b_k x^k + CJ_n(x)\log x. \tag{5.79}$$

As in the previous examples, the coefficients b_k can be determined from equation $L(y_2) = 0$, and we thus obtain

$$\begin{aligned} y_2(x) = b_0 x^{-n} + b_0 x^{-n} \sum_{j=1}^{n-1} \frac{1}{2^{2j} j!(n-1)\cdots(n-j)} x^{2j} - \frac{Ck_0}{2} s_n x^n - \\ - \frac{C}{2} \sum_{i=1}^{\infty} k_{2i} (s_i + s_{i+n}) x^{n+2i} + CJ_n(x)\log x. \end{aligned} \tag{5.80}$$

In the previous relation

$$\begin{aligned} s_m &= 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \\ k_{2i} &= \frac{(-1)^i}{2^{2i+n} i!(i+n)!}, \quad C = -\frac{b_0}{2^{n-1}(n-1)!}. \end{aligned} \tag{5.81}$$

In the special case, when $C = 1$, for b_0 we obtain

$$b_0 = -2^{n-1}(n-1)!, \tag{5.82}$$

and thus y_2 becomes

$$\begin{aligned} y_2 = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{x}{2}\right)^{2j} - \\ - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{1}{n!} \left(\frac{x}{2}\right)^n + J_n(x)\log x - \\ - \frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{i=1}^{\infty} \frac{(-1)^i}{i!(i+n)!} \left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{i}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{i+n}\right)\right] \left(\frac{x}{2}\right)^{2i}. \end{aligned} \tag{5.83}$$

Definition

The function K_n defined by the relation

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \\ -\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\frac{1}{n!}\left(\frac{x}{2}\right)^{n} + J_n(x)\log x - \\ -\frac{1}{2}\left(\frac{x}{2}\right)^{n}\sum_{i=1}^{\infty}\frac{(-1)^i}{i!(i+n)!}\times \\ \times\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{i}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{i+n}\right)\right]\left(\frac{x}{2}\right)^{2i}. \tag{5.84}$$

is called the **Bessel function of the second type of order** n.

Some Bessel functions

Let us now write the expressions for some Bessel functions

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots \tag{5.85}$$

$$J_1(x) = \frac{x}{2}\left(1 - \frac{x^2}{2\cdot 2^2} + \frac{x^4}{2\cdot 4^2\cdot 6} - \frac{x^6}{2\cdot 4^2\cdot 6^2\cdot 8} + \ldots\right) \tag{5.86}$$

Let us determine the functions $J_{n+1/2}$, where n is an integer. First, from (5.74), we determine $J_{1/2}$ and $J_{-1/2}$

$$J_{1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+\frac{1}{2}}}{k!\Gamma(\frac{3}{2}+k)}. \tag{5.87}$$

Further, as according to (5.126) and (5.138)

$$\Gamma(\tfrac{3}{2}+k) = \frac{1\cdot 3\cdot 5\ldots(2k+1)}{2^{k+1}}\Gamma(\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5\ldots(2k+1)}{2^{k+1}}\sqrt{\pi}, \tag{5.88}$$

by substituting into (5.87) we obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\sum_{k=0}^{\infty}\frac{(-1)^k\cdot x^{2k+1}}{(2k+1)!}. \tag{5.89}$$

As this sum represents the expansion of the function $\sin x$ into a series, we finally obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\sin x. \tag{5.90}$$

In a similar way, for $J_{-1/2}$ we obtain

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\cos x. \tag{5.91}$$

Now, according to (5.103), we can also calculate

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\left(-\cos x + \frac{\sin x}{x}\right) = \\ = \sqrt{\frac{2}{\pi\cdot x}}\left[\sin(x-\pi/2) + \frac{1}{x}\cos(x-\pi/2)\right],$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\left\{-\sin x + \frac{3}{x}\left[\sin(x-\pi/2) + \frac{1}{x}\cos(x-\pi/2)\right]\right\} = \\ = \sqrt{\frac{2}{\pi\cdot x}}\left[\left(1-\frac{3}{x^2}\right)\sin(x-\pi) + \frac{3}{x}\cos(x-\pi)\right]. \tag{5.92}$$

The last relation can be generalized for the purpose of calculating the Bessel function of the form $J_{n+1/2}$

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi\cdot x}}\left[P_n\left(\frac{1}{x}\right)\sin\left(x-\frac{n\pi}{2}\right) + Q_n\left(\frac{1}{x}\right)\cos\left(x-\frac{n\pi}{2}\right)\right] \tag{5.93}$$

where P_n and Q_n are polynomials of $1/x$.

From the last relation we can see that the Bessel function $J_{n+1/2}$ can be approximated by the expression

$$J_\nu(x) = \sqrt{\frac{2}{\pi\cdot x}}\left[\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) + O(x^{-1})\right], \quad x>0. \tag{5.94}$$

This asymptotic relation is valid not only for $\nu = n+1/2$, but also for every other ν.

There are tables of values of Bessel functions at specific points.

5.4.2 Weber functions

Let us now find the general solution for Bessel equations. To that end we shall introduce a new function defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}. \tag{5.95}$$

It can be proved that this function is a solution of the initial equation (5.45), as a linear combination of its solutions (the principle of superposition) when n is an integer. In that case the right hand side is an undetermined expression $0/0$. From this expression, by using L'Hospitals rule, we obtain, for an integer ($\nu = n$)

$$Y_n(x) = \frac{2}{\pi}J_n(x)\ln\frac{x}{2} - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \\ -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k-n}}{k!(k+n)!}\left[\frac{\Gamma'(k+1)}{\Gamma(k+1)} + \frac{\Gamma'(n+k+1)}{\Gamma(n+k+1)}\right]. \tag{5.96}$$

The function Y_n is called the **Weber**[9] **function**. The Weber function is a solution of Bessel equation in the case when ν is an integer ($\nu = n$).

[9] Heinrich Weber (1842-1913), German mathematician.

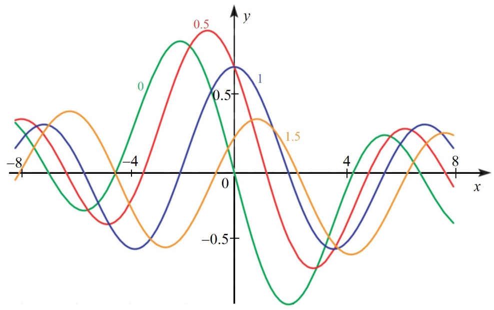

Figure 5.4: Weber function.

In the special case, for $n=0$ we obtain

$$Y_0(x)=\frac{2}{\pi}J_0(x)\ln\frac{x}{2}-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k}}{(k!)^2}\frac{\Gamma'(k+1)}{\Gamma(k+1)}. \tag{5.97}$$

The functions J_ν and Y_ν are linearly independent, and for every ν (integer or not integer) they form the fundamental solution of the initial equation. The general solution can now be represented in the form

$$y=C_1J_\nu(x)+C_2Y_\nu(x), \tag{5.98}$$

where $C_i\,(i=1,2)$ are arbitrary constants.

Let us now write some recurrence formulas for the Bessel and Weber functions.

$$J_\nu'(x)=J_{\nu-1}(x)-\frac{\nu}{x}J_\nu(x), \tag{5.99}$$

$$Y_\nu'(x)=Y_{\nu-1}(x)-\frac{\nu}{x}Y_\nu(x), \tag{5.100}$$

$$J_\nu'(x)=-J_{\nu+1}(x)+\frac{\nu}{x}J_\nu(x), \tag{5.101}$$

$$Y_\nu'(x)=-Y_{\nu+1}(x)+\frac{\nu}{x}Y_\nu(x), \tag{5.102}$$

$$J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x)-J_{\nu-1}(x), \tag{5.103}$$

$$Y_{\nu+1}(x)=\frac{2\nu}{x}Y_\nu(x)-Y_{\nu-1}(x). \tag{5.104}$$

These formulas can be verified by differentiating the Bessel and Weber functions. Let us demonstrate this on the example (5.99)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\nu J_\nu(x)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k2(\nu+k)x^{2\nu+2k-1}}{2^{\nu+2k}k!\Gamma(\nu+k+1)} \tag{5.105}$$

Further, as $\Gamma(\nu+k+1)=(\nu+k)\Gamma(\nu+k)$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\nu J_\nu(x)\right)=x^\nu\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+\nu-1}}{k!\Gamma(\nu-1+k+1)}, \tag{5.106}$$

and according to (5.74)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\nu J_\nu(x)\right)=x^\nu J_{\nu-1}. \tag{5.107}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\nu}J_{\nu}(x)\right) = \nu x^{\nu-1}J_{\nu}(x) + x^{\nu}J'_{\nu}(x). \tag{5.108}$$

From here it follows

$$x^{\nu}J_{\nu-1} = \nu x^{\nu-1}J_{\nu}(x) + x^{\nu}J'_{\nu}(x),$$

and then, by dividing with x^{ν} we obtain (5.99).

5.5 Some other special functions

Almost without exception, the most commonly used special functions are trigonometric (Fourier series), hyperbolic, Bessel and Legendre functions. However, there are several notable problems in physics and engineering, the solution of which imposes the introduction of some other functions. In this chapter we will only list a group of these functions, without going into details and analyzing their properties.

5.5.1 Hermit polynomials

The function denoted by $\mathrm{He}_n(x)$, which represents the solution of the differential equation

$$y'' - xy' + ny = 0, \tag{5.109}$$

is given by the expression

$$\mathrm{He}_n(x) = x^n - \frac{n!}{2!(n-2)!}x^{n-2} + 1\cdot 3\frac{n!}{4!(n-4)!}x^{n-4} - 1\cdot 3\cdot 5\frac{n!}{6!(n-6)!}x^{n-6} + \cdots \tag{5.110}$$

Functions defined in this way are called **Hermit polynomials**[10].

These polynomials can also be represented by the following relation

$$\mathrm{He}_n(x) = (-1)^n e^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2/2}\right), \quad n = 0, 1, \ldots \tag{5.111}$$

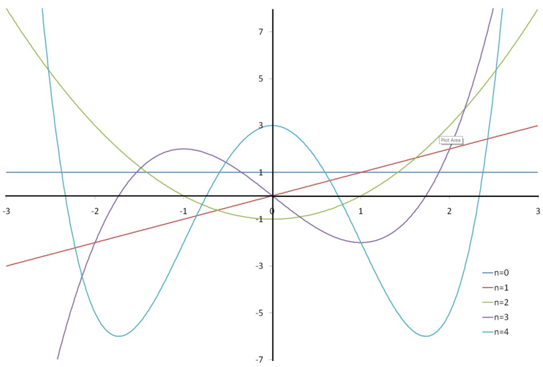

Figure 5.5: Hermit polynomials.

Some recurrence formulas

$$\begin{aligned} \mathrm{He}_{n+1}(x) &= x\mathrm{He}_n(x) - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{He}_n(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{He}_n(x) &= n\mathrm{He}_{n-1}(x). \end{aligned} \tag{5.112}$$

[10] Charles Hermite (1822-1901), Frencn mathematician, known for his work in algebra and number theory.

A relation between the exponential and Hermit functions

$$e^{tx-t^2/2} = \sum_{n=0}^{\infty} \mathrm{He}_n(x)\frac{t^n}{n!}. \tag{5.113}$$

Representation by integral

$$\mathrm{He}_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+it)^n e^{-t^2/2}\,\mathrm{d}t, \quad i=\sqrt{-1}. \tag{5.114}$$

Note that in the literature the equation

$$y'' - 2xy' + 2ny = 0$$

is often called Hermit differential equation, with the solution given by

$$\mathrm{H}_n^*(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right), \quad n = 0,1,\ldots$$

5.5.2 Laguerre polynomials

the solution of the differential equation

$$xy'' + (\alpha+1-x)y' + ny = 0 \tag{5.115}$$

is the function of the form

$$Ln^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}x^{n+\alpha}\right), \quad n = 0,1,\ldots, \tag{5.116}$$

which is called the **Laguerre polynomial**[11] (function).

5.6 Special functions that are not a result of the Frobenius method

In addition to the previously introduced special functions, which appeared as a result of solving differential equations, we will mention additional important functions that appear in solving some problems of mathematics and physics.

5.6.1 Gamma function (factorial function)

Definition

Γ function is defined by the following relation

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\,\mathrm{d}t, z \in \mathbb{C}, \tag{5.117}$$

where $t^{z-1} = e^{(z-1)\ln t}$.

[11] Edmond Laguerre (1834-1886), French mathematician, known for his work in geometry and infinite series theory.

It is easy to show that the above integral converges for all complex values z for which $\mathrm{Re}(z) > 0$. Namely,

$$\begin{aligned}\Gamma(x+iy) &= \int_0^\infty e^{-t}\, t^{x-1+iy}\,\mathrm{d}t = \int_0^\infty e^{-t}\, t^{x-1}\, e^{iy\ln t}\,\mathrm{d}t = \\ &= \int_0^\infty e^{-t}\, t^{x-1}\,[\cos(y\ln t) + i\sin(y\ln t)]\,\mathrm{d}t\end{aligned} \tag{5.118}$$

The expression in the square brackets is bounded for each t (as sin and cos are bounded functions), and thus the convergence, when $t \to \infty$, is provided by the term e^{-t}, while for the convergence in zero it is necessary to ensure that $x = \mathrm{Re}(z) > 1$.

A more general definition of the gamma function can often be found in literature.

Definition

Gamma function Γ is a meromorphic function of the independent variable z (without the additional condition $\mathrm{Re}(z) > 0$) so that its reciprocal value is

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-z/k}, \tag{5.119}$$

where $\gamma = 0,57721566\ldots$ is the Euler constant.

Definition

Euler constant γ is determined by the following expression

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0,5772156649. \tag{5.120}$$

This function satisfies the **reduction formula**,, that is, the following is valid

$$\Gamma(z+1) = z\Gamma(z), \quad \mathrm{Re}(z) > 0 \tag{5.121}$$

which can be proved by partial integration

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z\,\mathrm{d}t = \left[-e^{-t}\, t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}\, t^{z-1}\,\mathrm{d}t = z\Gamma(z). \tag{5.122}$$

In the special case, when the variable z is a real number ($z = \mathbb{R}(z)$), the Gamma function is determined by the expression

$$\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\,\mathrm{d}x, \quad n > 0. \tag{5.123}$$

This function is also known as the **Euler integral of the second kind**.

Specially, for $n = 1$, from (5.123), we obtain

$$\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = 1. \tag{5.124}$$

By partial integration, from (5.123) we obtain

$$\Gamma(n) = \cancelto{0}{\left[-e^{-x}x^{n-1}\right]_0^\infty} + (n-1)\underbrace{\int_0^\infty e^{-x}x^{n-2}\,\mathrm{d}x}_{\Gamma(n-1)}, \tag{5.125}$$

and then for $n > 1$

$$\Gamma(n) = (n-1)\cdot\Gamma(n-1). \tag{5.126}$$

Substituting n by $n+1$ yields $(n = 1, 2, \dots)$

$$\Gamma(n+1) = n\cdot\Gamma(n) = n\cdot(n-1)\cdot\Gamma(n-1) = \cdots = n! \tag{5.127}$$

If we substitute x by x^2 in (5.123), we obtain

$$\Gamma(n) = \int_0^\infty e^{-x^2}\cdot x^{2n-2}\,\mathrm{d}(x^2) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,\mathrm{d}x. \tag{5.128}$$

From here, for $n = 1/2$, it follows

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\,\mathrm{d}x. \tag{5.129}$$

Let us now show that the integral

$$\int_0^{\pi/2} \cos^m\tau\cdot\sin^n\tau\,\mathrm{d}\tau, \tag{5.130}$$

can be expressed in terms of the Γ – function.

Let us start from the integral

$$u = \int_0^\infty\int_0^\infty e^{-x^2-y^2}\cdot x^{2m-1}\cdot y^{2n-1}\,\mathrm{d}x\mathrm{d}y. \tag{5.131}$$

This double integral can be represented as a product of two single integrals

$$u = \int_0^\infty e^{-x^2}\cdot x^{2m-1}\,\mathrm{d}x\cdot\int_0^\infty e^{-y^2}\cdot y^{2n-1}\,\mathrm{d}y = \frac{1}{4}\Gamma(m)\cdot\Gamma(n). \tag{5.132}$$

On the other hand, by switching to polar coordinates

$$(x = r\cos\varphi,\ y = r\sin\varphi,\ \mathrm{d}x\mathrm{d}y = r\,\mathrm{d}r\mathrm{d}\varphi),$$

the integral (5.131) becomes

$$
\begin{aligned}
u &= \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}(r\cos\varphi)^{2m-1}\cdot(r\sin\varphi)^{2n-1}\, r\,\mathrm{d}r\mathrm{d}\varphi = \\
&= \int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\,\mathrm{d}r \int_0^{\pi/2}(\cos\varphi)^{2m-1}\cdot(\sin\varphi)^{2n-1}\,\mathrm{d}\varphi = \\
&= \frac{1}{2}\Gamma(m+n)\int_0^{\pi/2}(\cos\varphi)^{2m-1}\cdot(\sin\varphi)^{2n-1}\,\mathrm{d}\varphi
\end{aligned}
\tag{5.133}
$$

From (5.132) and (5.133) we obtain

$$
u = \frac{1}{4}\Gamma(m)\Gamma(n) = \frac{1}{2}\Gamma(m+n)\int_0^{\pi/2}(\cos\varphi)^{2m-1}\cdot(\sin\varphi)^{2n-1}\,\mathrm{d}\varphi, \tag{5.134}
$$

that is

$$
\int_0^{\pi/2}(\cos\varphi)^{2m-1}\cdot(\sin\varphi)^{2n-1}\,\mathrm{d}\varphi = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{5.135}
$$

If we now introduce the following substitutions

$$
\begin{aligned}
2m-1 = m' \quad &\Rightarrow \quad m = \frac{m'+1}{2}, \\
2n-1 = n' \quad &\Rightarrow \quad n = \frac{n'+1}{2},
\end{aligned}
\tag{5.136}
$$

the integral (5.135) becomes

$$
\int_0^{\pi/2}(\cos\varphi)^{m'}(\sin\varphi)^{n'}\,\mathrm{d}\varphi = \frac{1}{2}\frac{\Gamma\left(\dfrac{m'+1}{2}\right)\cdot\Gamma\left(\dfrac{n'+1}{2}\right)}{\Gamma\left(\dfrac{m'+n'+2}{2}\right)}. \tag{5.137}
$$

Note that $m' > -1$ and $n' > -1$, which follows from the condition that $m > 0$ and $n > 0$.

In the special case, when $m' = n' = 0$, we obtain

$$
\int_0^{\pi/2}\mathrm{d}\varphi = \frac{1}{2}\frac{[\Gamma(1/2)]^2}{\Gamma(1)} \quad \Rightarrow \quad \frac{\pi}{2} = \frac{[\Gamma(1/2)]^2}{2} \quad \Rightarrow \quad \underline{\Gamma(1/2) = \sqrt{\pi}}. \tag{5.138}
$$

Further, from (5.126), we obtain:

$$
\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \frac{1\cdot 3}{2^2}\sqrt{\pi}, \quad \Gamma(7/2) = \frac{1\cdot 3\cdot 5}{2^3}\sqrt{\pi}, \tag{5.139}
$$

etc.

From (5.127) it follows that

$$
\Gamma(n) = \frac{\Gamma(n+1)}{n}, \tag{5.140}
$$

and then $\Gamma(n) \to \infty$, when $n \to +0$.

Based on (5.140), the Γ – function can be extended for $n<0$, starting with the interval $(-1,0)$, followed by the interval $(-2,-1)$, etc. The function thus extended is depicted in Fig. 5.8.

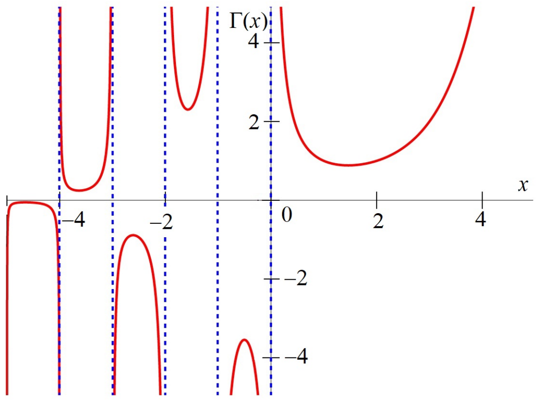

Figure 5.6: Γ function.

Also, using the relation (5.119), the gamma function can also be extended to the left half-plane, that is, for the values $\text{Re}(z) \le 0$

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \quad \text{Re}(z) > -n, \quad n \in \mathbb{N}, \quad z \in \mathbb{Z}_0^- := \{0,-1,-2,\ldots\} \tag{5.141}$$

where $(z)_n$ for $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ is defined by

$$(z)_0 = 1, \ (z)_n = z(z+1)\cdots(z+n-1), \tag{5.142}$$

The previous relations yield the following identities

$$\Gamma(n+1) = (1)_n = n!, \tag{5.143}$$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{5.144}$$

Definition

For $z \in \mathbb{C}\backslash\{0,-1,-2,-3,\cdots\}$ **Euler Gamma function** is defined by

$$\Gamma(z+1) = \begin{cases} \int\limits_0^\infty e^{-t} t^{z-1} \,\mathrm{d}t, & \text{if } \text{Re}(z) > 0 \\ \Gamma(z+1)/z, & \text{if } \text{Re}(z) \le 0, \ z \ne 0,-1,-2,\ldots \end{cases} \tag{5.145}$$

that is, the function is defined in the entire complex plane except for zero and points having a negative integer value.

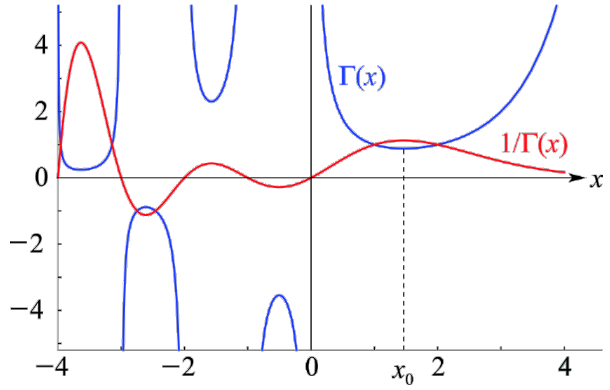

Figure 5.7: Euler $\Gamma(z)$ function (blue) and its reciprocal function (red) in the interval $[-4,4]$.

Properties of Euler gamma function

Euler gamma function has the following properties

1) for $\mathrm{Re}(z) \le 0$ the integral $\int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t$ is equivalent to the following expression

$$\Gamma(z) = \int_0^1 \left(\ln\left(\frac{1}{t}\right)\right)^{z-1} \mathrm{d}t, \tag{5.146}$$

2) for $z \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$ the following **reduction formula** is valid

$$\Gamma(z+1) = z\Gamma(z), \tag{5.147}$$

3) for $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)!, \tag{5.148}$$

4) for $z \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$ the following formula is valid

$$\Gamma(1-z) = z\Gamma(-z). \tag{5.149}$$

5) The gamma function can be expressed in terms of the limit value of the following expression

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}, \quad \mathrm{Re}(z) \le 0. \tag{5.150}$$

The previous representation is equivalent to Euler's infinite product

$$\frac{1}{z}\prod_{n=1}^{\infty} \frac{(1+(1/n))^z}{1+(z/n)}. \tag{5.151}$$

6) Let $z \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$, then Euler gamma function can be defined by the following expression

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^{-z/k}, \tag{5.152}$$

where γ is the Euler constant. This expression is also known as the **Weierstrass**[12]**definition** of gamma function.

7) Euler gamma function is analytical for all $z \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$.
8) Euler gamma function is always different from zero.

9) For all non-integer values $z \in \mathbb{C}$ the following is true

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{i} \quad \Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}. \tag{5.153}$$

This relation is known as the **reflection theorem**.

10) For half-integer arguments $\Gamma(n/2)$, $n \in \mathbb{N}$ has a special form

$$\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}} \tag{5.154}$$

where $n!!$ denotes the double factorial, defined by

$$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1, & n>0, \quad \text{odd} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2, & n>0, \quad \text{even} \\ 1, & n = 0,-1. \end{cases} \tag{5.155}$$

The proof of these properties can be found in [38].

For different values of parameter α, from (5.154), we obtain values for the gamma function shown in the following table

α	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$
$\Gamma(\alpha)$	$\sqrt{\pi}$	$\frac{1}{2}\sqrt{\pi}$	$\frac{3}{2}\sqrt{\pi}$	$\frac{5}{2}\sqrt{\pi}$	$\frac{7}{2}\sqrt{\pi}$	$\frac{9}{2}\sqrt{\pi}$	$\frac{11}{2}\sqrt{\pi}$

Table 5.1: Some values of the gamma function.

Similarly, bearing in mind the definition of the gamma function, the generalized binomial coefficients can be defined as follows

Definition

Generalized binomial coefficients $\binom{\alpha}{k}$ for $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_0$, are defined by

$$\binom{\alpha}{k} = \frac{(-1)^{k-1}\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \tag{5.156}$$

Combining the ninth and the fourth property we obtain

$$\Gamma(-z)\Gamma(z+1) = \frac{\Gamma(1-z)}{-z}\Gamma(z)z = -\Gamma(1-z)\Gamma(z) = -\frac{\pi}{\sin(\pi z)}, \tag{5.157}$$

or more generally, for $k \in \mathbb{N}_0$

$$(-1)^{k+1}\Gamma(z-k)\Gamma(k+1-z) = \Gamma(-z)\Gamma(z+1). \tag{5.158}$$

As already pointed out, the gamma function can be expressed as

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{5.159}$$

[12] Weierstrass

This function has poles of the first order and can further be expressed by the following formula

$$\Gamma(z) = \frac{(-1)^k}{(z+k)k!}\left[1 + O(z+k)\right], \quad z \to k, \quad k \in \mathbb{N}_0, \tag{5.160}$$

which is obtained by the substitutions $z = 1-n$, $n-1 = k$. The coefficient $(z+k)^{-1}$ in the vicinity of the pole $z = -k$ is called the **residue** of the gamma function $\mathrm{Res}\Gamma(z) = \frac{(-1)^k}{k!}$. Here $f(z) = O(g(z))_{z\to a}$ denotes a function, where for $\varepsilon > 0$ for which $|z-a| < \varepsilon \quad \Rightarrow \quad \left|\frac{f(z)}{g(z)}\right| < M$, for some $M < \infty$.

The **Euler Psi function** is also defined as

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{5.161}$$

and **Legendre formula** as

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)(z+1/2) \tag{5.162}$$

as well as the generalized **Gauss-Legendre multiplication formula**

$$\Gamma(mz) = \frac{m^{mz-1/2}}{2\pi^{(m-1)/2}}\prod_{k=0}^{m-1}\Gamma(z+k/m), \quad m = 2,3,\ldots \tag{5.163}$$

Stirling formula

$$\Gamma(2z) = \sqrt{2\pi}\, z^{z-1/2}\, e^{-z}\left[1 + O(1/z)\right], \quad |\arg z| < \pi, \quad |z| \to \infty \tag{5.164}$$

and its consequences

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[1 + O(1/n)\right], \quad n \to \infty \tag{5.165}$$

$$|\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-1/2}\, e^{-\pi|y|/2}\left[1 + O(1/y)\right], \quad y \to \infty. \tag{5.166}$$

5.6.2 Beta function

Definition

Beta function is defined by the following relation

$$B(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,\mathrm{d}x \tag{5.167}$$

for each $m > 0$ and $n > 0$. This condition is necessary for convergence of the integral.

The function (5.167) is also known as **Euler integral of the first kind**. Beta function can be related to Γ – function, starting from (5.167) and introducing the substitution $x = \cos^2\varphi$, which yields (according to (5.135)):

$$B(m,n) = 2\int_0^{\pi/2}(\cos\varphi)^{2m-1}(\sin\varphi)^{2n-1}\,\mathrm{d}\varphi = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = B(m,n). \tag{5.168}$$

On this basis, the definition can now be broadened to the set of complex numbers.

Definition

Beta function $B(z,\omega)$ with respect to two variables $z,\omega \in \mathbb{C}$ is defined by

$$B(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}. \tag{5.169}$$

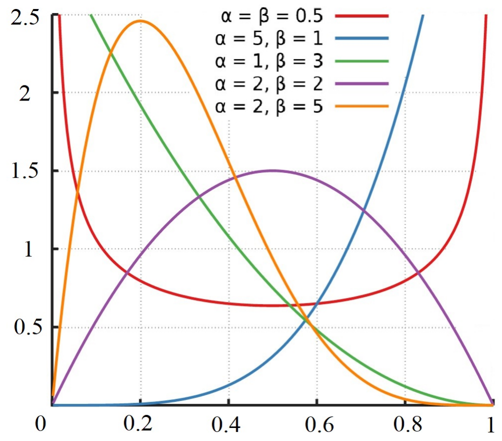

Figure 5.8: Beta function.

Properties of Beta function

Beta function has the following properties.

a) For $\mathrm{Re}(z) > 0$, $\mathrm{Re}(\omega) > 0$ the equation (5.169) is equivalent to

$$\begin{aligned} B(z,\omega) &= \int_0^1 t^{z-1}(1-t)^{\omega-1}\,\mathrm{d}t = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+\omega}}\,\mathrm{d}t = \\ &= 2\int_0^{\pi/2} (\sin t)^{2z-1}(\cos t)^{2\omega-1}\,\mathrm{d}t. \end{aligned} \tag{5.170}$$

b) $B(z+1,\,\omega+1)$ is a solution of the Beta integral

$$\int_0^1 t^z(1-t)^\omega\,\mathrm{d}t = B(z+1,\,\omega+1). \tag{5.171}$$

c) The following identities stand

$$\begin{aligned} B(z,\omega) &= B(\omega,z), \\ B(z,\omega) &= B(z+1,\omega)+B(z,\omega+1), \\ B(z,\omega+1) &= \frac{\omega}{z}B(z+1,\omega) = \frac{\omega}{z+\omega}B(z,\omega). \end{aligned} \tag{5.172}$$

5.6.3 Error function

Definition

The following integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t \tag{5.173}$$

defines a function called the **error function**.

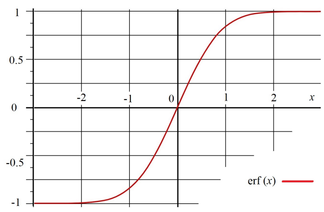

Figure 5.9: Error function.

This function can be represented as a series

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)k!}. \tag{5.174}$$

Definition

A **complementary error function** or erfc function is also used, defined by the relation

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, \mathrm{d}t. \tag{5.175}$$

From the definitions of these functions and (5.173) and (5.174) it follows that

$$\operatorname{erf}(\infty) = 1 \quad \text{and} \quad \operatorname{erfc}(0) = 1. \tag{5.176}$$

Definition

Functions $\mathrm{C}(x)$ and $\mathrm{S}(x)$ defined by the following relations

$$\begin{aligned} \mathrm{C}(x) &= \int_0^x \cos\frac{\pi t^2}{2}\, \mathrm{d}t, \\ \mathrm{S}(x) &= \int_0^x \sin\frac{\pi t^2}{2}\, \mathrm{d}t, \end{aligned} \tag{5.177}$$

are called **Fresnel**[13]**integrals**.

In some problems of physics, the function of the following form appears

$$\frac{1}{1+i}\operatorname{erf}\left(\frac{1+i}{2}x\sqrt{\pi}\right) = \mathrm{C}(x) + i\mathrm{S}(x), \quad i = \sqrt{-1}. \tag{5.178}$$

where C(x) and S(x) are Fresnel integrals.

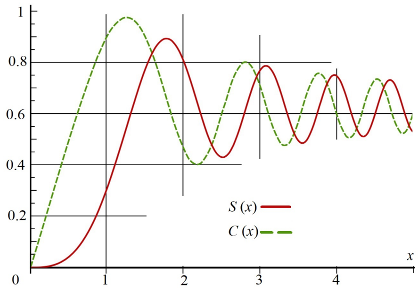

Figure 5.10: Fresnel integrals.

5.6.4 Exponential integrals

The integral given by the relation

$$-\mathrm{Ei}(-x) = \int\limits_x^\infty \frac{e^{-t}}{t}\,\mathrm{d}t \tag{5.179}$$

defines the so called **exponential integral**. This function also appears in many problems of physics.

For small values of x this integral can be approximated by the relation

$$-\mathrm{Ei}(x) \approx -\gamma - \ln x, \tag{5.180}$$

where γ is a constant, given by the relation (5.120).

If x is substituted by iy, the exponential integral can be represented in the following form

$$\mathrm{Ei}(iy) = \mathrm{Ci}(y) + i\mathrm{Si}(y) + i\frac{\pi}{2}, \tag{5.181}$$

where two new functions, Ci(y) and Si(y), have been introduced, defined by the following expressions

$$\begin{aligned} \mathrm{Ci}(y) &= -\int\limits_y^\infty \frac{\cos t}{t}\,\mathrm{d}t = \gamma + \ln y - \int\limits_0^y \frac{1-\cos t}{t}\,\mathrm{d}t, \\ \mathrm{Si}(y) &= \int\limits_0^y \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} - \int\limits_y^\infty \frac{1-\sin t}{t}\,\mathrm{d}t. \end{aligned} \tag{5.182}$$

These functions are called Ci(y) – **cosine integral** and Si(y) – **sine integral**. The constant γ is Euler's constant.

[13]Augustin Fresnel (1788-1827), French physicist, known for his work in optics.

5.7 Mittag-Leffler functions

It is known that the solution of a linear differential equation with constant coefficients can be represented as an exponential function. On the other hand, the fractional differential equation with constant coefficients has in many cases a solution given in the form of the so-called **Mittag-Leffler function**. It is not difficult to see, as will be shown later, that the Mittag-Leffler function is a generalization of the exponential function.

The function introduced by Mittag-Leffler in 1903, which contains one parameter and which today bears his name, can be considered a generalization of the exponential function, because it is reduced to it when the parameter takes the value one. Mittag-Leffler functions will be presented here as functions of a real argument, as well as of two or three parameters.

Definition (Mittag-Leffler function)

The Mittag-Leffler function of parameter α, $Re(\alpha) > 0$, also known as the classical Mittag-Leffler function, denoted by $E_\alpha(x)$, is defined by

$$E_\alpha(x) := \sum_{k=0} \frac{x^k}{\Gamma(\alpha k + 1)}. \tag{5.183}$$

An alternative definitions is

$$E_\alpha(x^\alpha) := \sum_{k=0} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

In both cases, for $\alpha = 1$ the exponential function is obtained, and thus both definitions can be considered fractional generalizations of the exponential function.

There are several generalizations of the Mittag-Leffler function, taking the number of parameters as the number of variables.

Definition (Two-parameter Mittag-Leffler function)

Let $x \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{C}$ be two parameters, where $Re(\alpha) > 0$ and $Re(\beta) > 0$. The **two-parameter Mittag-Leffler function** is defined by

$$E_{\alpha,\beta}(x) := \sum_{k=0} \frac{x^k}{\Gamma(\alpha k + \beta)}. \tag{5.184}$$

This two-parameter Mittag-Leffler function is a generalization of the function defined by the previous expression, as for $\beta = 1$ we obtain $E_{\alpha,1}(x) = E_\alpha(x)$.

Definition (Three-parameter Mittag-Leffler function)

Let $x \in \mathbb{C}$ and $\alpha, \beta, \rho \in \mathbb{C}$ be three parameters, where $Re(\alpha) > 0$, $Re(\beta) > 0$ and $Re(\rho) > 0$. The **three-parameter Mittag-Leffler function** is defined as

$$E^\rho_{\alpha,\beta}(x) := \sum_{k=0} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{x^k}{k!}, \tag{5.185}$$

where $(\rho)_k$ denotes the so-called **Pochhammer symbol** defined by

$$(\rho)_n = \rho(\rho+1)\cdots(\rho+n-1) = \frac{\Gamma(\rho+n)}{\Gamma(\rho)}, n \geq 0, (\rho)_0 \quad = 1. \tag{5.186}$$

In the case when $\rho = 1$ it follows that $E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$.

Shukla and Prajapati [68] introduce the Mittag-Leffler function with four parameters defined as follows:

Definition (Mittag-Leffler function with four parameters)

Mittag-Leffler function with four parameters is defined by the series

$$E^{\rho,q}_{\alpha,\beta}(z) := \sum_{k=0} \frac{(\rho)_{qk}}{\Gamma(k\alpha b)k!} \frac{x^k}{k!}, \tag{5.187}$$

where $Re(\alpha) > 0$, $Re(\rho) > 0$ and $(\rho)_{qk} = \frac{\Gamma(\rho+qk)}{\Gamma(\rho)}$ ăre four parameters.

R Generally speaking, the exponential function e^z, Mittag-Leffler function $E_\alpha(z)$, Wiman function $E_{\alpha,\beta}(z)$ and Prabhakar function $E^\rho_{\alpha,\beta}(z)$ are special cases of Shukla function $E^{\rho,q}_{\alpha,\beta}(z)$.

Relation between two Mittag-Leffler functions

Let $x \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ be two parameters, where $Re(\alpha) > 0$, $Re(\beta) > 0$ and $k = 1, 2, \ldots$. The following relation between two Mittag-Leffler functions is true

$$\frac{d^k}{dx^k} E_{\alpha,\beta}(x) = k! E^{k+1}_{\alpha,\beta+\alpha k}(x). \tag{5.188}$$

We shall now introduce two other functions that appear in different practical problems, related to Mittag-Leffler functions.

Definition (Miller-Ross function)

The function

$$E_x(\nu,\alpha) = x^\nu \sum_{k=0}^{\infty} \frac{(ax)^k}{\Gamma(\nu+k+1)} \tag{5.189}$$

where $x \in \mathbb{R}$, $Re(\nu) > -1$ and $a \in \mathbb{R}$, is called the **Miller-Ross function**.

Definition (Rabotnov function)

The function

$$R_\alpha(\beta, x) = x^\alpha \sum_{k=0}^{\infty} \frac{(\beta^k x)^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)} \tag{5.190}$$

where $Re(\alpha) > -1$ i $\beta \in \mathbb{R}$, is called the **Rabotnov function**.

5.8 Elliptic integrals

It can be observed that the introduction of inverse operations in Mathematics, for example for numbers, leads to numbers of a "new nature".

Let's illustrate this with a few examples:

- addition - *inverse* - subtraction, leads to negative numbers,
- multiplication - *inverse* - division, leads to fractions,
- power of a number - *inverse* - root of a number, leads to irrational numbers,
- derivative - *inverse* - integral, leads to results that cannot be expressed in terms of elementary functions, but rather by functional relations between variables.

Let us illustrate the last item by several examples. Observe the following integrals, and the corresponding solutions

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \log\left(x+\sqrt{1+x^2}\right)+C,$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C.$$

These integrals can be observed as special cases of the following integrals

$$J_1 = \frac{\mathrm{d}x}{\sqrt{P_2+(x)}},\qquad J_2 = \frac{\mathrm{d}x}{\sqrt{P_2-(x)}} \tag{5.191}$$

where

$$P_2+(x) = ax^2+bx+c, \quad a>0,$$
$$P_2+(x) = ax^2+bx+c, \quad a<0,$$

are second order polynomials. For these polynomials the values of the integrals are

$$J_1 = \frac{1}{\sqrt{a}}\log(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c})+C,$$
$$J_2 = \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax+b}{\sqrt{b^2-4ac}}+C.$$

We can see that for relatively simple forms of sub-integral functions

$$1/\sqrt{1\pm x^2}$$

we obtain relatively complex functions (*logarithm* and the inverse trigonometric function - *arcsine*).

Examples in which roots of polynomials of the third and fourth degree appear are, of course, more complex and lead to new functions. Such functions (integrals) are obtained, for example, when calculating the length of the arc of an ellipse. The solution of this problem leads to integrals of the form

$$\int R(x,\sqrt{P})\,\mathrm{d}x$$

where R is a rational function, and P is a third or fourth degree polynomial. These integrals are called **elliptic integrals**.

These types of integrals can be reduced to three basic elliptic integrals

$$\int \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad \int \frac{z^2\,\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad \int \frac{\mathrm{d}z}{(1+hz^2)\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{5.192}$$

where k and h are constants, and h can be an imaginary value, which Legendre named elliptic integrals of the first, second and third kind.

The constant k is called the **elliptic modulus**, and takes its values in the interval $0<k<1$.

The substitution $z=\sin\varphi$ yields the so called **Legendre elliptic integrals** of the first, second and third kind

$$\begin{aligned} F(\varphi,k) &= \int \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \\ E(\varphi,k) &= \int \sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi, \\ \Pi(n,\varphi,k) &= \int \frac{\mathrm{d}\varphi}{(1+h\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}. \end{aligned} \tag{5.193}$$

These integrals are called **normal** (incomplete), if they are functions of the upper boundary φ, or **complete**, if the upper boundary $\varphi=\pi/2$.

$$\begin{aligned} F(\varphi,k) &= \int_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^{x} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad x=\sin\varphi,\ k^2<1, \\ E(\varphi,k) &= \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi, = \int_0^{x} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad x=\sin\varphi,\ k^2<1, \\ \Pi(n,\varphi,k) &= \int_0^{\varphi} \frac{\mathrm{d}\varphi}{(1+h\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} = \int_0^{x} \frac{1}{1+ht^2}\frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \end{aligned} \tag{5.194}$$

For the first two integrals Legendre even produced tables with two arguments: independent variable φ and parameter k, that is, φ and θ, if the substitution $k=\sin\theta$ is introduced. The angles are expressed in degrees.

Solutions of many practical problems lead to these integrals. Besides the already mentioned determining of the length of the arc of an ellipse, let us also mention the task of movement of mathematical and spherical pendulum (see Examples 214 and 215, p. 297 and 300).

5.8.1 Some properties of the integral $F(\varphi,k)$

1. $F(0,k)=0$, which is a direct consequence of the definition ($\int_0^0=0$)

2.

$$F(\varphi+m\pi,k)=F(\varphi,k)+mF(\pi,k). \tag{5.195}$$

Proof (for $m=1$)

$$\begin{aligned} F(\varphi+\pi,k) &= \int_0^{\varphi+\pi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^{\pi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \int_{\pi}^{\varphi+\pi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \\ &= \int_0^{\pi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \int_0^{\theta} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \\ &= F(\varphi,k)+F(\pi,k). \end{aligned}$$

3. Legendre normal elliptic integral of the first order is an odd function

$$F(-\varphi,k) = -F(\varphi,k) \tag{5.196}$$

Proof.

$$\int_0^{-\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

by substitution $-\varphi = \theta$ becomes

$$-\int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = -F(\varphi,k)$$

From (5.195), and according to (5.196), we obtain $\varphi = -\pi/2$

$$F\left(\frac{\pi}{2},k\right) = F\left(-\frac{\pi}{2},k\right) + F(\pi,k) = -F\left(\frac{\pi}{2},k\right) + F(\pi,k)$$

and from that

$$F(\pi,k) = 2F\left(\frac{\pi}{2},k\right).$$

Definite integral (usually denoted by K)

$$K = F\left(\frac{\pi}{2},k\right) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \tag{5.197}$$

is called a **complete elliptic integral of the first kind**.

5.8.2 Elliptic functions

Using the property (5.195) we obtain the following relation

$$F(\varphi+n\pi,k) = F(\varphi,k) + 2nK, \tag{5.198}$$

where n is a positive or negative integer.

The variable φ in equation (5.289), taken as a function of λt obtained by an inversion of the integral, namely

$$\varphi = \varphi(\lambda,t) \tag{5.199}$$

represents the **amplitude** of the normal elliptic integral of the first order

$$\varphi = \operatorname{am}\lambda t \tag{5.200}$$

and it is called the **Jacobi elliptic function**. It has a dimension of angle.

5.8.3 Complete elliptic integrals of the first and second kind

$$
\begin{aligned}
K &= K(k) = F(\pi/2,k) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right),\\
K(\alpha) &= \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\sin^2\alpha\sin^2\varphi}},\\
E(k) &= E(\pi/2,k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi == \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right),\\
E(\alpha) &= \int_0^{\pi/2} \sqrt{1-\sin^2\alpha\sin^2\varphi}\,\mathrm{d}\varphi
\end{aligned}
\tag{5.201}
$$

where $F(\pm\frac{1}{2},\frac{1}{2};1;k^2)$ is the **Gaussian hypergeometric function**.

Complementary integrals

By introducing the substitution $k' = \sqrt{1-k^2}$, where k' is the **complementary modulus**, and k the modulus of the elliptic integral, we obtain the following values

$$
\begin{aligned}
K'(k) &= K(k') = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-(1-k^2)\sin^2\varphi}} = F(\pi/2,k'),\\
E'(k) &= E(k') = \int_0^{\pi/2} \sqrt{1-(1-k^2)\sin^2\varphi}\,\mathrm{d}\varphi = E(\pi/2,k')
\end{aligned}
\tag{5.202}
$$

It should be emphasized that in this case $()'$ does not denote a derivative!!!

Legendre relation

$$K\cdot E' + EK' - KK' = \frac{\pi}{2}$$

5.8.4 Jacobi elliptic functions

Jacobi elliptic functions are inverse elliptic integrals. If $u(\varphi,k)$ (elliptic integral of the first kind) is given by

$$u = \int_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^{\sin\varphi} \frac{\mathrm{d}t}{(10t^2)\sqrt{1-k^2t^2}} \tag{5.203}$$

for $k^2 < 1$, then the inverse function is

$$\varphi = \mathrm{am}\,u \quad \text{(amplitude of } u\text{)}. \tag{5.204}$$

$$
\begin{aligned}
\mathrm{sn}\,u &= \mathrm{sn}(u,k) = \sin\phi = \sin(\mathrm{am}\,u),\\
\mathrm{cn}\,u &= \mathrm{cn}(u,k) = \cos\phi = \cos(\mathrm{am}\,u) = \sqrt{1-\mathrm{sn}^2 u},\\
\mathrm{dn}\,u &= \mathrm{dn}(u,k) = \sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\mathrm{sn}^2(u)}
\end{aligned}
\tag{5.205}
$$

Note that

$$u = \int_1^{\mathrm{cn}(u,k)} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(k'^2+k^2t^2)}},$$

$$u = \int_1^{\mathrm{dn}(u,k)} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(t^2-k'^2)}}.$$

5.8.5 Main properties of elliptic functions

Relations between elliptic functions

Akin to trigonometric functions, the following relations exist for elliptic functions

$$\begin{aligned}\mathrm{sn}^2 v + \mathrm{cn}^2 v &= 1,\\ \mathrm{dn}^2 v + k^2\,\mathrm{sn}^2 v &= 1,\\ \mathrm{dn}^2 v - k^2\,\mathrm{cn}^2 v &= k^2.\end{aligned} \tag{5.206}$$

The relations follow directly from the definitions of these functions.

Some values

$$\begin{array}{llll} \mathrm{sn}(0,k)=0, & \mathrm{cn}(0,k)=1, & \mathrm{dn}(0,k)=1, & \mathrm{am}(0,k)=0,\\ \mathrm{sn}(u,0)=\sin u, & \mathrm{cn}(u,0)=\cos u, & \mathrm{dn}(u,0)=1, & \\ \mathrm{sn}(u,1)=\mathrm{th}u, & \mathrm{cn}(u,1)=\mathrm{sech}u, & \mathrm{dn}(u,1)=\mathrm{sech}u. & \end{array} \tag{5.207}$$

Symmetricity of elliptic functions

$$\mathrm{su}(-u) = -\mathrm{su}(u), \quad \mathrm{cu}(-u) = \mathrm{cu}(u), \tag{5.208}$$

$$\mathrm{du}(-u) = \mathrm{du}(u), \quad \mathrm{am}(-u) \ = -\mathrm{am}(u). \tag{5.209}$$

Additivity formulas

$$\begin{aligned}\mathrm{sn}(u\pm v) &= \frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(u)\pm\mathrm{cn}(u)\,\mathrm{sn}(v)\,\mathrm{dn}(u)}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}\\ \mathrm{cn}(u\pm v) &= \frac{\mathrm{cn}(u)\,\mathrm{cn}(v)\mp\mathrm{sn}(u)\mathrm{dn}(u)\,\mathrm{sn}(v)\,\mathrm{dn}(v)}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}, \quad \mathbf{??}\\ \mathrm{dn}(u\pm v) &= \frac{\mathrm{dn}(u)\,\mathrm{dn}(v)\mp k^2\,\mathrm{sn}(u)\mathrm{cn}(u)\,\mathrm{sn}(v)\,\mathrm{cn}(v)}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}.\end{aligned} \tag{5.210}$$

Some properties and periodicity of elliptic functions

The main properties of elliptic functions follow directly from equations (5.196) and (5.198):

$$\begin{aligned}\mathrm{am}(-v) &= -\mathrm{am}\,v,\\ \mathrm{am}(v+2nK) &= \mathrm{am}\,v + n\pi.\end{aligned} \tag{5.211}$$

It can be seen that this function is odd, but that it is not "purely" periodic, but rather pseudoperiodic. From (5.200) and $u=\sin\varphi$ the following basic elliptic functions are obtained

$$\begin{aligned}\sin\varphi &= \sin(\mathrm{am}\,v) = \mathrm{sn}\,v,\\ \cos\varphi &= \cos(\mathrm{am}\,v) = \mathrm{cn}\,v,\\ \sqrt{1-k^2\sin(\mathrm{am}\,v)} &= \mathrm{dn}\,v.\end{aligned} \tag{5.212}$$

Elliptic functions are double periodic with regard to the variable u:

$$\begin{aligned}
&\mathrm{sn}(u,k) \quad \text{periodic} \quad 4K \quad \text{and} \quad 2iK', \\
&\mathrm{cn}(u,k) \quad \text{periodic} \quad 4K \quad \text{and} \quad 2K+2iK', \\
&\mathrm{dn}(u,k) \quad \text{periodic} \quad 2K \quad \text{and} \quad 4iK'.
\end{aligned} \tag{5.213}$$

This is not hard to prove. For example, according to (5.211) it follows that

$$\begin{aligned}
\mathrm{sn}(v+4K) &= \sin \mathrm{am}(v+4K) = \sin(\mathrm{am}(v+2\pi)), \\
&= \sin \mathrm{am}\, v = \sin \mathrm{sn}\, v
\end{aligned}$$

and similarly for the remaining functions.

These three elliptic functions were also introduced by Jacobi, while the short denotement was introduced by Gudermann[14]. Beside the real period $4K$ these functions have also and imaginary period. They are thus duble periodical.

Derivatives of elliptic functions

$$\begin{aligned}
\frac{\mathrm{d}(\mathrm{sn}\, v)}{\mathrm{d}v} &= \frac{\mathrm{d}\sin\varphi}{\mathrm{d}v} = \cos\varphi\,\frac{\mathrm{d}\varphi}{\mathrm{d}v} = \cos\varphi\,\frac{1}{\frac{\mathrm{d}v}{\mathrm{d}\varphi}} = \\
&= \cos\varphi\sqrt{1-k^2\sin^2\varphi} = \mathrm{cn}\, v\,\mathrm{dn}\, v, \\
\frac{\mathrm{d}(\mathrm{cn}\, v)}{\mathrm{d}v} &= -\mathrm{sn}\, v\,\mathrm{dn}\, v, \\
\frac{\mathrm{d}(\mathrm{dn}\, v)}{\mathrm{d}v} &= -k^2\mathrm{sn}\, v\,\mathrm{cn}\, v.
\end{aligned} \tag{5.214}$$

Integrals of elliptic functions

$$\begin{aligned}
\int \mathrm{sn}(u)\,\mathrm{d}u &= \frac{1}{k}\left(\mathrm{dn}(u) - k\,\mathrm{cn}(u)\right), \\
\int \mathrm{cn}(u)\,\mathrm{d}u &= \frac{1}{k}\cos^{-1}(\mathrm{dn}(u)), \\
\int \mathrm{dn}(u)\,\mathrm{d}u &= \mathrm{am}(u) = \sin^{-1}(\mathrm{sn}(u)).
\end{aligned} \tag{5.215}$$

Expansion of elliptic functions into series

$$\begin{aligned}
\mathrm{sn}(u,k) &= u - (1+k^2)\frac{u^3}{3!} + \left(1+14k^2+k^4\right)\frac{u^5}{5!} + \left(1+135k^2+135k^4+k^6\right)\frac{u^7}{7!} + \cdots \\
\mathrm{cn}(u,k) &= 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - (1+44k^2+16k^4)\frac{u^6}{6!} + \cdots \\
\mathrm{dn}(u,k) &= 1 - k^2\frac{u^2}{2!} + k^2(4+k^2)\frac{u^4}{4!} - k^2(16+44k^2+k^4)\frac{u^6}{6!} + \cdots
\end{aligned} \tag{5.216}$$

If the following substitutions are introduced

$$q = e^{-piK/K'} \quad \text{and} \quad v = \pi u/(2K),$$

[14]Gudermann

we obtain

$$\begin{aligned} \operatorname{sn}(u,k) &= \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}}\sin\left[(2n+1)\,v\right], \\ \operatorname{cn}(u,k) &= \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}\cos\left[(2n+1)\,v\right], \\ \operatorname{dn}(u,k) &= \frac{\pi}{K}+\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n}}{1+q^{2n}}\cos(2n\,v). \end{aligned} \tag{5.217}$$

Determining the value of the elliptic integral K

The function $\frac{1}{\sqrt{1-k^2\sin^2\varphi}}$ can be expanded into a series (following the binomial formula), which yields

$$\frac{1}{\sqrt{1-k^2\sin^2\varphi}} = 1+\frac{1}{2}k^2\sin^2\varphi+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\varphi+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6\sin^6\varphi+\cdots$$

This series is uniformly convergent for $k^2<1$. According to the Wallis[15] formula

$$\int_0^{\pi/2}\sin^{2n}\varphi\,\mathrm{d}\varphi = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{2}$$

and we obtain

$$K=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\cdots\right]. \tag{5.218}$$

5.9 Orthogonal and normalized functions

Observe the set of integrable functions for $x\in[a,b]$, $(a<b)$

$$f_1(x),\, f_2(x),\ldots,\, f_n(x),\ldots \tag{5.219}$$

Definition

The function set (5.219) is said to be **orthogonal** in the interval $[a,b]$ if

$$(f_m,\, f_n)\int_a^b f_m(x)\,f_n(x)\,\mathrm{d}x = 0, \quad \text{for } \forall\, m\neq n,\; n,m=1,2,\ldots \tag{5.220}$$

where functions $f_n(x)$, $n=1,2,\ldots$, are not identically equal to zero in the observed interval.

As $f_n(x)$ are integrable functions, and a and b are constants, then the following integral obviously also exists

$$\int_a^b f_n^2(x)\,\mathrm{d}x = I_n > 0,$$

where I_n is constant.

[15] Wallis

Definition

Non-negative square root

$$\sqrt{(f_n, f_n)} = \sqrt{\int\limits_a^b f_n^2(x)\,\mathrm{d}x} = \sqrt{I_n} \tag{5.221}$$

is called the **norm** of the function $f_n(x)$ and denoted by

$$\|f_n\| = \sqrt{I_n}, \quad \text{i.e.} \quad \|f_n\| = \sqrt{(f_n, f_n)} = \sqrt{\int\limits_a^b f_n^2(x)\,\mathrm{d}x}. \tag{5.222}$$

Definition

The function set f_n (5.219), whose norm is equal to one, i.e.

$$\|f_n\| = \sqrt{\int\limits_a^b f_n^2(x)\,\mathrm{d}x = 1} \tag{5.223}$$

is called a **normalized** function set.

Definition

The function set f_n (5.219) that is both orthogonal and normalized, i.e.

$$(f_m, f_n) = \int\limits_a^b f_m(x) f_n(x)\,\mathrm{d}x = \delta_{mn} \tag{5.224}$$

is called an **orthonormal** function set on the interval $x \in [a, b]$.

In the previous relation δ_{ij} represents Kronecker delta symbol.

Some function sets, important for applications, are not orthogonal but have the following property

$$\int\limits_a^b p(x) f_m(x) f_n(x)\,\mathrm{d}x = 0, \quad \text{for } m \neq n. \tag{5.225}$$

In this case it is said that the function set f_n (5.219) is **orthogonal to the weight function** $p(x)$, on the interval $x \in [a, b]$.

The norm of functions from this set is defined by the following expression

$$\|f_n\| = \sqrt{\int\limits_a^b p(x) f_n^2(x)\,\mathrm{d}x}. \tag{5.226}$$

If this norm is equal to one, then the function set is **orthonormal with respect to** $p(x)$ on the observed interval.

5.9.1 Series of orthogonal functions

A significant type of functional series is introduced by means of orthogonal function sets in a simple way. Namely, let $g_1(x), g_2(x), \ldots$, be a set of orthogonal functions on the interval $a \leq x \leq b$, and let $f(x)$ be a given function that can be represented on this interval by a convergent series

$$f(x) = \sum_{n=1}^{\infty} a_n g_n(x). \tag{5.227}$$

Then this series is called a **generalized Fourier series**[16] of the function $f(x)$, and its coefficients $a_1, a_2, \ldots$, are called **Fourier coefficients** of the function $f(x)$ with respect to the given orthogonal function set. Given the orthogonality of the functions g_i, the Fourier coefficients can be determined relatively easily. Multiplying the left and right side of the equality (5.227) by $g_m(x)$, and then integrating from a to b (assuming that element-by-element integration is possible), we obtain

$$(f, g_m) = \int_a^b f g_m dx = \int_a^b \left(\sum_{n=1}^{\infty} a_n g_n \right) g_m dx = \sum_{n=1}^{\infty} a_n \left(\int_a^b g_n g_m dx \right) = \sum_{n=1}^{\infty} a_n (g_n, g_m)$$

For $n = m$ we obtain $(g_m, g_m) = \|g_m\|^2$, while for $n \neq m$, due to the orthogonality of functions g_i, $(g_n, g_m) = 0$. Thus, the formula for Fourier coefficients is

$$a_n = \frac{(f, g_n)}{\|g_n\|^2} = \frac{1}{\|g_n\|^2} \int_a^b f(x) g_n(x) dx, \qquad n = 1, 2, \ldots$$

5.9.2 Completeness of orthonormal functions

In practice, orthonormal sets are often used with a "sufficient number" of functions to allow generalized Fourier series of these functions to represent broad classes of functions, for example, all continuous functions on the interval $a \leq x \leq b$.

Definition

A function sequence $f_n(x)$ is **convergent with respect to norm** and converges to function f if

$$\lim_{n \to \infty} \|f_n - f\| = 0, \tag{5.228}$$

that is, if (omitting the square root of the norm)

$$\lim_{n \to \infty} \int_a^b [f_n(x) - f(x)]^2 dx = 0.$$

[16] Fourier series will be discussed in more detail in the next chapter.

Convergence with respect to norm is also called **mean-square convergence** or **mean convergence**. According to this definition the series (5.227) converges (with respect to norm) to function f if

$$\lim_{n\to\infty}\int_a^b [s_n(x)-f(x)]^2\,dx=0,$$

where $s_n(x)$ is the partial sum of the series (5.227)

$$s_n(x)=\sum_{k=1}^{n} a_k g_k(x).$$

Definition

The set of orthonormal functions $g_1, g_2, \ldots$ is **complete** in the set of functions S on the interval $a \le x \le b$, if any function f from S can be approximated with arbitrary accuracy by the linear combination $a_1 g_1 + a_2 g_2 + \cdots + a_n g_n$. This means that for each $\varepsilon > 0$ constants $a_1, a_2, \ldots, a_n$ can be found, such that for a sufficiently large n

$$\| f-(a_1 g_1 + a_2 g_2 + \ldots + a_n g_n)\| < \varepsilon.$$

It can be shown that the sets of Legendre polynomials and Bessel functions are complete in the set of continuous real functions on appropriate intervals.

5.9.3 Sturm–Liouville problem

In engineering, various important orthogonal function sets appear as solutions of second-order linear differential equations of the form

$$\left[r(x)y'\right]' + [q(x)+\lambda p(x)]\,y=0, \tag{5.229}$$

on an interval $a \le x \le b$, with the following boundary conditions

$$\begin{array}{ll} a) & k_1\,y(a)+k_2\,y'(a)=0,\\ b) & l_1\,y(b)+l_2\,y'(b)=0. \end{array} \tag{5.230}$$

Here λ is a parameter and k_i, that is, l_i $(i=1,2)$, are given (known) real constants, which are not both equal to zero.

The equation (5.229) is called the **Sturm**[17] **– Liouville**[18] **equation**.

It can be shown that Legendre, Bessel and some other equations can be represented in this form.

The problem of solving the differential equation (5.229) with boundary conditions (5.230) is called the **Sturm–Liouville problem**.

[17] Jacques Charles Francois Sturm (1803-1855), French mathematician of Swiss origin. He made a significant contribution to algebra, and is known for being the first to calculate the speed of sound in water.

[18] Joseph Liouville (1809-1882), French mathematician. He made important contributions in various fields of mathematics, and his work in complex analysis, special functions, differential geometry and number theory is especially well known.

Principal values. Principal functions

From relations (5.229) and (5.230) it can be seen that for every λ, there exists a trivial solution $y \equiv 0$, i.e. $y(x) = 0$ for $\forall x$ from the observed interval.

Definition

If there exists a value λ, for which the problem (5.229), (5.230) has a non-trivial solution ($y \not\equiv 0$), this value is called the **principal value** of the problem.

Definition

The non-trivial solution of the problem (5.229), (5.230), which corresponds to the principal value λ is called the **principal function**.

We will give some properties of the previously introduced concepts in the form of two (following) theorems.

Theorem 16

Assume that the functions p, q, r i r' in equation (5.229) are real an continuous on the interval $a \le x \le b$. Let $y_m(x)$ and $y_n(x)$ be two principal functions of the Sturm–Liouville problem (5.229), (5.230), which correspond to two different principal values λ_m and λ_n, respectively. Then y_m and y_n are orthogonal functions on the observed interval, with respect to the weight function p.

Proof

Given that y_m and y_n are solutions of the observed problem, they satisfy the following relations

$$\left(ry_m'\right)' + (q+\lambda_m p)y_m = 0,$$
$$\left(ry_n'\right)' + (q+\lambda_n p)y_n = 0.$$

If the first relation is multiplied by y_n, and the second relation by $-y_m$, and then the two resulting relations are added, we obtain

$$(\lambda_m - \lambda_n)\, p y_m y_n = y_m \left(ry_n'\right)' - y_n \left(ry_m'\right)' = \left[\left(ry_n'\right) y_m - \left(ry_m'\right) y_n\right]'. \tag{5.231}$$

This expression represents a continuous function on the interval $a \le x \le b$, because r and r' are continuous functions according to the initial assumption, and y_m and y_n are also continuous functions, as solutions of the problem. Thus, we can integrate the observed expression (5.231), which yields

$$\begin{aligned}(\lambda_m - \lambda_n) \int_a^b p y_m y_n \,\mathrm{d}y &= \left[r\left(y_n' y_m - y_m' y_n\right)\right]\Big|_a^b = \\ &= r(b)\left[y_n'(b)y_m(b) - y_m'(b)y_n(b)\right] - r(a)\left[y_n'(a)y_m(a) - y_m'(a)y_n(a)\right].\end{aligned} \tag{5.232}$$

Let us now analyze the expression on the left side of equation (5.232), and to that end let us observe the boundary conditions (5.230):

$$k_1 y_m(a) + k_2 y_m'(a) = 0, \tag{5.233}$$
$$k_1 y_n(a) + k_2 y_n'(a) = 0. \tag{5.234}$$

Multiplying the first equation by y_n and the second by y_m, and then subtracting the resulting equations, we obtain

$$k_2\left[y_m(a)y_n'(a) - y_n(a)y_m'(a)\right] = 0. \tag{5.235}$$

Assuming that $k_2 \neq 0$ we obtain

$$y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0. \tag{5.236}$$

Similarly, it can be shown that also

$$y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0, \tag{5.237}$$

for $l_2 \neq 0$. Based on these relations we conclude that

$$\int_a^b p y_m y_n \, \mathrm{d}y = 0, \quad \text{for } m \neq n. \tag{5.238}$$

We have thus proved the theorem for $k_2 \neq 0$ and $l_2 \neq 0$.

Let us observe again conditions (5.233) and (5.234). By multiplying the first condition by y_n' and the second by y_m', and then subtracting the resulting equations, we obtain, for $k_1 \neq 0$

$$y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0. \tag{5.239}$$

Similarly, for $l_1 \neq 0$ we obtain

$$y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0. \tag{5.240}$$

Thus the theorem is proved for this case as well, and given that k_1 and k_2, that is, l_1 and l_2 can not be both equal to zero, the theorem is proved in its entirety.

Theorem 17

If the Sturm–Liouville problem (5.229), (5.230) satisfies the conditions of the previous theorem, and if $p \neq 0$ on the entire interval $a \leq x \leq b$, then all principal values of the problem are real.

Proof

Let us assume the contrary, namely that $\lambda = \alpha + i\beta$ is a principal value of the problem, and that the corresponding principal function has the form

$$y(x) = u(x) + iv(x).$$

In these expressions α and β are real constants, and u and v real functions.

By substituting these values into equation (5.229) we obtain

$$\left(ru' + irv'\right)' + (q + \alpha p + i\beta p)(u + iv) = 0.$$

In order for this complex equation to be satisfied, it is necessary that both its real and imaginary parts be equal to zero, i.e.

$$\begin{aligned} \left(ru'\right)' + (q + \alpha p)\, u - \beta p v &= 0, \\ \left(rv'\right)' + (q + \alpha p)\, v + \beta p u &= 0. \end{aligned}$$

If we multiply the first equation by v, and the second equation by $-u$, and then add the resulting equations, we obtain

$$\begin{aligned} -\beta \left(u^2 + v^2\right) p = u(rv')' - v(ru')' &= \\ &= \left[(rv')u - (ru')v\right]'. \end{aligned}$$

The expression in the square brackets is a continuous function on the interval $a \leq x \leq b$ (see the proof of the previous theorem), and thus, by integration, taking into account the boundary conditions (as in the case of the previous theorem), we obtain

$$-\beta \int_a^b \left(u^2 + v^2\right) p \,\mathrm{d}x = \left[r\left(uv' - vu'\right)\right]\Big|_a^b = 0.$$

Given that y is a principal function, it follows that $y \not\equiv 0$. Further, as y and p are continuous functions, where $p > 0$ or $p < 0$ on the interval $a \leq x \leq b$, and $y^2 = u^2 + v^2 \not\equiv 0$, it follows that the integral on the left side of the last equation is not equal to zero. From this, it follows that β must be equal to 0, i.e. $\beta = 0$. Given that $\lambda = \alpha + i\beta$ and $\beta = 0$, it follows that $\lambda = \alpha$, that is, λ is a real number. Thus the theorem has been proved.

5.10 Examples

Problem 186

Prove that the function set

$$1,\ \sin x,\ \sin 2x,\ \ldots,\ \sin nx,\ \ldots, \tag{5.241}$$

is orthogonal on the interval $[-\ell, \ell]$.

Solution

We shall use here the following, well known trigonometric relations

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right),$$
$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right),$$
$$\sin\alpha\cdot\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right).$$

It can be easily proved that the values of the following integrals are

$$\int_{-\ell}^{\ell} \sin\frac{k\pi x}{\ell}\cdot\sin\frac{m\pi x}{\ell}\,\mathrm{d}x = \begin{cases} 0, & k\neq m,\\ \ell, & k=m\neq 0,\\ 0, & k=m=0\end{cases}$$

$$\int_{-\ell}^{\ell} \cos\frac{k\pi x}{\ell}\cdot\cos\frac{m\pi x}{\ell}\,\mathrm{d}x = \begin{cases} 0, & k\neq m,\\ \ell, & k=m\neq 0,\\ 2\ell, & k=m=0\end{cases}$$

$$\int_{-\ell}^{\ell} \cos\frac{k\pi x}{\ell}\cdot\sin\frac{m\pi x}{\ell}\,\mathrm{d}x = 0$$

Let us show this on the first example, only. For $x\in[-\ell,\ell]$, if $k\neq m$

$$\int_{-\ell}^{\ell} \sin\frac{k\pi x}{\ell}\cdot\sin\frac{m\pi x}{\ell}\,\mathrm{d}x = \frac{1}{2}\int_{-\ell}^{\ell}\left[\cos\frac{(k-m)\pi x}{\ell} - \cos\frac{(k+m)\pi x}{\ell}\right]\mathrm{d}x =$$
$$= \frac{1}{2}\int_{-\ell}^{\ell}\cos\frac{(k-m)\pi x}{\ell}\,\mathrm{d}x - \frac{1}{2}\int_{-\ell}^{\ell}\cos\frac{(k+m)\pi x}{\ell}\,\mathrm{d}x =$$
$$= \frac{1}{2}\frac{\ell}{(k-m)\pi}\sin\frac{(k-m)\pi x}{\ell}\bigg|_{-\ell}^{\ell} - \frac{1}{2}\frac{\ell}{(k+m)\pi}\sin\frac{(k+m)\pi x}{\ell}\bigg|_{-\ell}^{\ell} =$$
$$= \frac{\ell}{2(k-m)\pi}\sin\frac{(k-m)\pi}{\ell}2\ell - \frac{\ell}{2(k+m)\pi}\sin\frac{(k+m)\pi}{\ell}2\ell =$$
$$= \frac{\ell}{2(k-m)\pi}\sin 2(k-m)\pi - \frac{\ell}{2(k+m)\pi}\sin 2(k+m)\pi =$$
$$= 0.$$

When $k = m$ the previous integral becomes

$$\int_{-\ell}^{\ell} \sin\frac{k\pi x}{\ell}\cdot\sin\frac{k\pi x}{\ell}\,\mathrm{d}x = \int_{-\ell}^{\ell}\frac{1-\cos\frac{2k\pi x}{\ell}}{2}\,\mathrm{d}x =$$

$$= \left(\frac{x}{2} - \frac{\ell}{4k\pi}\sin\frac{2k\pi x}{\ell}\right)\Bigg|_{-\ell}^{\ell} = \ell.$$

We can see that this set is not normalized, as $\ell \neq 1$.

Problem 187

The function set

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos nx,\ \sin nx, \ldots, \tag{5.242}$$

is orthogonal, for $x \in [-\ell, \ell]$.

Solution

As in the previous case, it can be shown that the condition for orthogonality (5.220), p. 263, stands.

Problem 188

Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right],$$

are orthogonal on the interval $x \in [-1, +1]$.

Solution

To prove this, we will start from the differential equation satisfied by these polynomials

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0.$$

This equation can be rewritten in a form more suitable for future work

$$\left[\left(1-x^2\right)y'\right]' + n(n+1)y = 0.$$

If the polynomials $P_n(x)$ and $P_m(x)$ are solutions of this differential equation, they satisfy the following equations

$$\left[\left(1-x^2\right)P_n'\right]' + n(n+1)P_n = 0,$$

$$\left[\left(1-x^2\right)P_m'\right]' + m(m+1)P_m = 0.$$

Multiplying the first equation by $P_m(x)$, and the second equation by $P_n(x)$, we obtain

$$P_m\left[\left(1-x^2\right)P_n'\right]'+n(n+1)P_nP_m=0,$$
$$P_n\left[\left(1-x^2\right)P_m'\right]'+m(m+1)P_mP_n=0.$$

By subtracting the resulting equations we further obtain

$$[n(n+1)-m(m+1)]\,P_mP_n+P_m\left[\left(1-x^2\right)P_n'\right]'-P_n\left[\left(1-x^2\right)P_m'\right]'=0.$$

Let us note that

$$\begin{aligned}
&P_m\left[\left(1-x^2\right)P_n'\right]'-P_n\left[\left(1-x^2\right)P_m'\right]'=\\
&=\left[P_m\left(1-x^2\right)P_n'\right]'-P_m'\left(1-x^2\right)P_n'-\left[P_n\left(1-x^2\right)P_m'\right]'+P_n'\left(1-x^2\right)P_m'=\\
&=\left[P_n\left(1-x^2\right)P_m'-P_m\left(1-x^2\right)P_n'\right]'-\left(1-x^2\right)\left(P_n'P_m'-P_m'P_n'\right)=\\
&=\left[P_n\left(1-x^2\right)P_m'-P_m\left(1-x^2\right)P_n'\right]'.
\end{aligned}$$

Thus, the previous relation can be rewritten as

$$[n(n+1)-m(m+1)]\,P_mP_n+\left[\left(1-x^2\right)\left(P_nP_m'-P_mP_n'\right)\right]'=0.$$

Integrating this relation from -1 to $+1$ yields

$$\begin{aligned}
&[n(n+1)-m(m+1)]\int_{-1}^{+1}P_m(x)P_n(x)\,\mathrm{d}x+\\
&\qquad+\int_{-1}^{+1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-x^2\right)\left(P_nP_m'-P_mP_n'\right)\right]\mathrm{d}x=0,\quad \text{for } n\neq m.
\end{aligned}$$

Given that

$$\int_{-1}^{+1}\mathrm{d}\left[\left(1-x^2\right)\left(P_nP_m{}'-P_mP_n'\right)\right]=\left(1-x^2\right)\left(P_nP_m'-P_mP_n'\right)\Big|_{-1}^{+1}=0,$$

because, for $x=-1$ and $x=+1$, $1-x^2=0$, and $P_n(x)$, $P_n'(x)$, $P_m(x)$ and $P_m'(x)$ are bounded functions, it follows that $P_n(\pm1)<\infty$, $P_m(\pm1)<\infty$, $P_n'(\pm1)<\infty$ i $P_m'(\pm1)<\infty$, and, as $[n(n+1)-m(m+1)]\neq0$, we finally obtain

$$\int_{-1}^{+1}P_m(x)P_n(x)\,\mathrm{d}x=0,\quad \text{za } n\neq m,$$

which is the condition of orthogonality.

For the norm we obtain

$$\|P_n(x)\|^2=\frac{1}{(2^nn!)^2}\int_{-1}^{+1}\left[\frac{\mathrm{d}^nu_n(x)}{\mathrm{d}x^n}\right]^2\mathrm{d}x,$$

where

$$u_n(x) = \left(x^2-1\right)^n,$$

and also

$$u_n(\pm 1) = u_n'(\pm 1) = \cdots = u_n^{(n-1)}(\pm 1) = 0.$$

The previous integral, by way of partial integration, becomes

$$\int\limits_{-1}^{+1} \left[\frac{\mathrm{d}^n u_n(x)}{\mathrm{d}x^n}\right]\left[\frac{\mathrm{d}^n u_n(x)}{\mathrm{d}x^n}\right]\mathrm{d}x =$$

$$= \quad u_n^{(n)}(x)\cdot u_n^{(n-1)}(x)\Big|_{-1}^{+1} + (-1)\int\limits_{-1}^{+1}\left[\frac{\mathrm{d}^{n-1}u_n(x)}{\mathrm{d}x^{n-1}}\right]\left[\frac{\mathrm{d}^{n+1}u_n(x)}{\mathrm{d}x^{n+1}}\right]\mathrm{d}x \quad \Rightarrow$$

$$\int\limits_{-1}^{+1} \left[\frac{\mathrm{d}^n u_n(x)}{\mathrm{d}x^n}\right]\left[\frac{\mathrm{d}^n u_n(x)}{\mathrm{d}x^n}\right]\mathrm{d}x = (-1)^n \int\limits_{-1}^{+1} u_n(x)\frac{\mathrm{d}^{2n}u_n(x)}{\mathrm{d}x^{2n}}.$$

Given that

$$u_n = \left(x^2-1\right)^n = x^{2n} + \binom{n}{1}\left(x^2\right)^{n-1}(-1) + \binom{n}{2}\left(x^2\right)^{n-2}(-1)^2 + \cdots$$

for the derivative within the integral we obtain

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(u_n) = \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(x^{2n}\right) = 2n(2n-1)\cdots 2\cdot 1 = (2n)!.$$

The square of the norm can now be represented as

$$\|P_n(x)\|^2 = \frac{1}{(2^n n!)^2}(2n)!(-1)^n \int\limits_{-1}^{+1} u_n\,\mathrm{d}x.$$

However, given that

$$u_n = \left(x^2-1\right)^n = (-1)^n\left(1-x^2\right)^n,$$

the previous expression can be rewritten in the form

$$\|P_n(x)\|^2 = \frac{1}{(2^n n!)^2}(2n)!\int\limits_{-1}^{+1}\left(1-x^2\right)^n\mathrm{d}x.$$

The integral in this equation can be partially integrated several times[19], which yields

$$\int_{-1}^{+1}(1-x)^n(1+x)^n\,\mathrm{d}x =$$

$$= (1-x)^n\,\frac{(1+x)^{n+1}}{n+1}\bigg|_{-1}^{+1} + \frac{n}{n+1}\int_{-1}^{+1}(1-x)^{n-1}(1+x)^{n+1}\,\mathrm{d}x =$$

$$= \frac{n}{n+1}\int_{-1}^{+1}(1-x)^{n-2}(n-1)\frac{(1+x)^{n+2}}{n+2}\,\mathrm{d}x =$$

$$= \frac{n}{n+1}\,\frac{n-1}{n+2}\int_{-1}^{+1}(1-x)^{n-2}(1+x)^{n+2}\,\mathrm{d}x = \cdots =$$

$$= \frac{n}{n+1}\,\frac{n-1}{n+2}\cdots\frac{1}{2n}\int_{-1}^{+1}(1+x)^{2n}\,\mathrm{d}x = \frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots(2n)}\int_{-1}^{+1}(1+x)^{2n}\,\mathrm{d}x.$$

Given that

$$\int_{-1}^{+1}(1+x)^{2n}\,\mathrm{d}x = \frac{(1+x)^{2n+1}}{2n+1}\bigg|_{-1}^{+1} = \frac{2^{2n+1}}{2n+1},$$

for the square of the norm we finally obtain

$$\|P_n(x)\|^2 = \frac{1}{(2^n n!)^2}(2n)!\frac{n!1\cdot 2\cdots n}{1\cdot 2\cdots n\cdot(n+1)\cdots(2n)}\,\frac{2\cdot 2^{2n}}{2n+1} = \frac{2}{2n+1}.$$

We have thus proved that Legendre polynomials are orthogonal, but that they are not normalized.

Problem 189

Prove that for each fixed $n = 0, 1, \ldots$, Bessel functions $J_n(\lambda_{1n}x), J_n(\lambda_{2n}x), \ldots$, form a set of orthogonal functions on the interval $0 \le x \le R$, with respect to the weight function $p(x) = x$, i.e. that

$$\int_0^R xJ_n(\lambda_{kn}x)\,J_n(\lambda_{mn}x)\,\mathrm{d}x = \begin{cases} 0, & m \neq k, \\ \frac{R^2}{2}J^2_{n+1}(\lambda_{mn}R), & m = k. \end{cases} \tag{5.243}$$

Problem 190

Prove that

$$\int_{-\infty}^{+\infty} He_m(x)He_n(x)e^{-x^2/2}\,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ n!\sqrt{2\pi}, & n = m, \end{cases} \tag{5.244}$$

[19] Given that $\int u,\mathrm{d}v = uv - \int v,\mathrm{d}u$, we assumed that in this case $u = (1-x)^2$, and $\mathrm{d}v = (1+x)^n\,\mathrm{d}x$.

i.e. Hermite functions form a set of orthogonal functions, with respect to the weight function $p(x) = e^{-x^2/2}$.

Problem 191

Prove that Laguerre functions satisfy the following equations

a)

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) e^{-x}\, \mathrm{d}x = \delta_{mn}, \tag{5.245}$$

b)

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) e^{-x} x^\alpha\, \mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \dfrac{(n+\alpha)!}{n!}, & n = m. \end{cases} \tag{5.246}$$

Legendre polynomials

Problem 192

Prove that the function $G(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$ generates Legendre polynomials, i.e.

$$G(x,t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Solution

The observed function has two singularities $G(x,t)$, namely the two zeroes of the polynomial $1 - 2xt + t^2$:

$$t_{1,2} = x \pm i\sqrt{1 - x^2}.$$

As the absolute value of both solutions $|t_1| = |t_2| = 1$, we can conclude that the function can be expanded into Taylor series in the vicinity of point $t = 0$, as the series is convergent for $|t| < 1$.

If $|2xt - t^2| < 1$ we have the following expansion

$$\begin{aligned} (1 - 2xt + t^2)^{-\frac{1}{2}} &= \left[1 - (2xt - t^2)\right]^{-\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} (2xt - t^2)^k \\ &= \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} (2xt - t^2)^k \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} k!\, k!} (2xt - t^2)^k. \end{aligned} \tag{5.247}$$

If the condition

$$|t|(2|x| + |t|) < 1, \tag{5.248}$$

is satisfied, we can arbitrarily group the terms in relation (5.247).

Coefficient next to t^n in relation (5.247) is determined by equating it with the coefficient next to t^n in the expansion

$$\sum_{k=0}^{n} \frac{(2k)!}{2^{2k}\,k!\,k!}(2xt-t^2)^k,$$

that is

$$\sum_{k=0}^{n} \frac{(2n-2k)!}{2^{2n-2k}\,(n-k)!\,(n-k)!}(2xt-t^2)^{n-k}, \tag{5.249}$$

because

$$\sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} f(n-k).$$

From (5.249) the coefficient next to t^n is obtained in the form

$$\sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^{2n-2k}\,(n-k)!\,(n-k)!}\binom{n-k}{k}(2x)^{n-2k},$$

that is

$$\sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^{n}\,(n-k)!\,(n-k)!}\binom{n-k}{k}x^{n-2k},$$

where k takes the values from $k=0$ to $k=[n/2]$, because the binomial coefficient $\binom{n-k}{k}$ for $k>[n/2]$ is equal to zero. From this it follows that

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^{n}\,(n-k)!\,(n-k)!}\binom{n-k}{k}x^{n-2k}. \tag{5.250}$$

This proof has been developed under the assumption that $-1 \le x \le 1$. The function

$$G(x,t) = (1-2xt+t^2)^{-\frac{1}{2}} \tag{5.251}$$

is called the *generating function* of Legendre polynomial.

Problem 193

Prove the Bonnet recursive formula

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{5.252}$$

if the *Legendre polynomial* $P_n(x)$ is expressed in terms of the function $G(x,t)$

$$G(x,t) = \frac{1}{\sqrt{1-2xt-t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Solution

If we start from

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{5.253}$$

and the differentiate both the left an right hand side by t, we obtain the equality

$$-\frac{1}{2}\frac{-2x+2t}{\sqrt{(1-2xt+t^2)^3}} = \sum_{n=1}^{\infty} P_n(x)nt^{n-1},$$

and from there

$$\frac{x-t}{1-2xt+t^2}\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=1}^{\infty} P_n(x)nt^{n-1}.$$

Based on (5.253), and after multiplying by $1-2xt+t^2$, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} P_n(x)nt^{n-1},$$

that is

$$x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} =$$
$$= \sum_{n=1}^{\infty} P_n(x)nt^{n-1} - 2x\sum_{n=1}^{\infty} P_n(x)nt^n + \sum_{n=1}^{\infty} P_n(x)nt^{n+1}.$$

Reducing all powers in the sums to t^n we obtain

$$x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n =$$
$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)t^n - 2x\sum_{n=1}^{\infty} P_n(x)nt^n + \sum_{n=2}^{\infty} P_{n-1}(x)(n-1)t^n.$$

Separating the terms of the sum for $n=0$ and $n=1$ we obtain

$$xP_0 + xP_1t + x\sum_{n=2}^{\infty} P_n(x)t^n - P_0t - \sum_{n=2}^{\infty} P_{n-1}(x)t^n =$$
$$= P_1 + 2P_2t + \sum_{n=2}^{\infty}(n+1)P_{n+1}(x)t^n - 2xP_1t - 2x\sum_{n=2}^{\infty} nP_n(x)t^n +$$
$$+ \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n,$$

that is

$$xP_0 + 3xP_1t - P_0t - P_1 - 2P_2t =$$
$$= \sum_{n=2}^{\infty}(n+1)P_{n+1}(x)t^n - x\sum_{n=2}^{\infty}(2n+1)P_n(x)t^n + \sum_{n=2}^{\infty} nP_{n-1}(x)t^n.$$

As the three first Legendre polynomials are [35]

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \frac{1}{2}(3x^2-1),$$

it follows that

$$xP_0 + 3xP_1t - P_0t - P_1 - 2P_2t = x + 3x^2t - t - x - (3x^2 - 1)t = 0,$$

and from there

$$\sum_{n=2}^{\infty}(n+1)P_{n+1}(x)t^n - x\sum_{n=2}^{\infty}(2n+1)P_n(x)t^n + \sum_{n=2}^{\infty}nP_{n-1}(x)t^n = 0.$$

Grouping coefficients next to t^n we obtain

$$\sum_{n=2}^{\infty}[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)]t^n = 0.$$

from where it follows

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

Problem 194

Prove the Christoffel recurrence formula

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \tag{5.254}$$

if the *Legendre polynomial* $P_n(x)$ is defined by the expansion

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n.$$

Solution

If we start from

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n,$$

and then differentiate both the left and the right hand side by x, we obtain

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty}P'_n(x)t^n.$$

If we substitute the expression $(1-2xt+t^2)^{-1/2}$ in this relation by the sum from (5.253), and then multiply both sides by $(1-2xt+t^2)$, we obtain

$$t\sum_{n=0}^{\infty}P_n(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty}P'_n(x)t^n,$$

$$\sum_{n=0}^{\infty}P_n(x)t^{n+1} = \sum_{n=0}^{\infty}P'_n(x)t^n - 2x\sum_{n=0}^{\infty}P'_n(x)t^{n+1} + \sum_{n=0}^{\infty}P'_n(x)t^{n+2}$$

Reducing all to sums of t^{n+1} we obtain

$$\sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=-1}^{\infty} P'_{n+1}(x)t^{n+1} - 2x\sum_{n=0}^{\infty} P'_n(x)t^{n+1} + \sum_{n=1}^{\infty} P'_{n-1}(x)t^{n+1}$$

Given that these sums start from different values of n, we shall separate the terms for $n=-1$ and $n=0$, which yields

$$P_0 t + \sum_{n=1}^{\infty} P_n(x)t^{n+1} = P'_0 + P'_1 t + \sum_{n=1}^{\infty} P'_{n+1}(x)t^{n+1} - 2xP'_0 t - 2x\sum_{n=1}^{\infty} P'_n(x)t^{n+1} + \sum_{n=1}^{\infty} P'_{n-1}(x)t^{n+1}.$$

Substituting the Legendre polynomial $P_0 = 1$ and the derivatives of Legendre polynomials $P_0^{'} = 0$ i $P_1^{'} = 1$, we obtain

$$t + \sum_{n=1}^{\infty} P_n(x)t^{n+1} = 0 + t + \sum_{n=1}^{\infty} P'_{n+1}(x)t^{n+1} - 0 - 2x\sum_{n=1}^{\infty} P'_n(x)t^{n+1} + \sum_{n=1}^{\infty} P'_{n-1}(x)t^{n+1}.$$

From there, after grouping coefficients next to t^{n+1}, it follows

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x). \tag{5.255}$$

Further, by differentiating the Bonnet formula (equations 5.252) by x, we obtain

$$(n+1)P'_{n+1}(x) - (2n+1)xP'_n(x) - (2n+1)P_n(x) + nP'_{n-1}(x) = 0. \tag{5.256}$$

Eliminating P'_n from equations (5.255) and (5.256) we obtain the required recurrent formula.

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

Problem 195

Prove that Legendre polynomials satisfy the Legendre differential equation

$$(x^2-1)y^{''} + 2xy^{'} - n(n+1)y = 0.$$

Solution

If we start from

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

and then differentiate both sides by x, we obtain

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x)t^n.$$

Differentiating the initial equation by t yields

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n(x)nt^{n-1}.$$

Eliminating $(1-2xt+t^2)^{-3/2}$ from these two equations, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n'(x)t^n = t\sum_{n=0}^{\infty} P_n(x)nt^{n-1},$$

that is

$$x\sum_{n=0}^{\infty} P_n'(x)t^n - \sum_{n=0}^{\infty} P_n'(x)t^{n+1} = \sum_{n=0}^{\infty} P_n(x)nt^n,$$

or

$$x\sum_{n=0}^{\infty} P_n'(x)t^n - \sum_{n=1}^{\infty} P_{n-1}'(x)t^n = \sum_{n=0}^{\infty} P_n(x)nt^n.$$

From here, equating coefficients next to t^n yields

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x). \tag{5.257}$$

By differentiating the Bonnet formula (equation 5.252) we obtain

$$(n+1)P_{n+1}'(x) - (2n+1)xP_n'(x) + nP_{n-1}'(x) = (2n+1)P_n(x).$$

After eliminating $P_{n-1}'(x)$ from the two previous equations, we obtain

$$P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x),$$

that is, substituting $n+1$ with n,

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x).$$

Eliminating again $P_{n-1}'(x)$ from the last equation and equation (5.257) yields

$$(x^2-1)P_n'(x) - nxP_n(x) + nP_{n-1}(x) = 0.$$

By differentiating this equation we obtain

$$(x^2-1)P_n''(x) + (2-n)xP_n'(x) - nP_n(x) + nP_{n-1}'(x) = 0.$$

Eliminating once again $P_{n-1}'(x)$ form previous equation and equation (5.257), we finally obtain

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0,$$

which was to be proved.

Problem 196

Prove the orthogonality of Legendre polynomials starting from Rodrigues formula for Legendre polynomials.

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

(see [**35**]).

Solution

It should be proved that

$$I_{mn} = \int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0, \qquad \text{for } m \neq n.$$

Starting from Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \tag{5.258}$$

for the integral I_{mn} we obtain

$$I_{mn} = \int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x =$$

$$= \frac{1}{2^{n+m}n!m!}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m\mathrm{d}x.$$

Let us assume that $m < n$. By partial integration we obtain

$$I_{mn} = \frac{1}{2^{m+n}m!n!}\left[\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\Big|_{-1}^{1} - \right.$$

$$\left. - \int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}(x^2-1)^{m+1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\mathrm{d}x\right] =$$

$$= \frac{-1}{2^{m+n}m!n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}(x^2-1)^{m+1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\mathrm{d}x.$$

If we repeat partial integration another $n-1$ times we obtain

$$I_{mn} = \frac{(-1)^n}{2^{m+n}m!n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+n}}{\mathrm{d}x^{m+n}}(x^2-1)^m(x^2-1)^n\mathrm{d}x.$$

As, according to the assumption that $m < n$, it follows that $m+n > 2m$, and as $(x^2-1)^m$ is a polynomial of degree $2m$ it follows that

$$\frac{\mathrm{d}^{m+n}}{\mathrm{d}x^{m+n}}(x^2-1)^m \equiv 0,$$

that is

$$I_{mn}\left(=\int_{-1}^{1}P_m(x)P_n(x)\mathrm{d}x\right)=0 \qquad \text{for } m\neq n,$$

and the polynomials are thus orthogonal.

Problem 197

Prove that the following equation holds for Legendre polynomials

$$P_n(\cos\theta)=\sum_{k=0}^{n}\binom{-1/2}{n}\binom{-1/2}{n-k}\cos\left(2k-n\right)\theta.$$

Solution:

Let us start from the generating function of Legendre polynomials

$$G(x,t)=(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n.$$

Given that

$$2\cos\theta=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta=e^{i\theta}+e^{-i\theta},$$

a $e^{i\theta}\cdot e^{-i\theta}=1$, it follows that

$$1-2t\cos\theta+t^2=1-t(e^{i\theta}+e^{-i\theta})+t^2e^{i\theta}\cdot e^{-i\theta}=(1-te^{i\theta})(1-te^{-i\theta}).$$

If we now introduce the substitution $x=\cos\theta$ into the expression $(1-2xt+t^2)^{-\frac{1}{2}}$ we obtain

$$(1-2t\cos\theta+t^2)^{-\frac{1}{2}}=(1-te^{i\theta})^{-1/2}(1-te^{-i\theta})^{-1/2}.$$

For $|t|<1$ the following expansions hold

$$(1-te^{i\theta})^{-1/2}=\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}e^{in\theta}t^n=\sum_{n=0}^{\infty}a_nt^n,$$

$$(1-te^{-i\theta})^{-1/2}=\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}e^{-in\theta}t^n=\sum_{n=0}^{\infty}b_nt^n,$$

and from there it follows that

$$(1-2xt+t^2)^{-\frac{1}{2}}=\left(\sum_{n=0}^{\infty}a_nt^n\right)\left(\sum_{n=0}^{\infty}b_nt^n\right).$$

Coefficients next to t^n in this expansion can be represented by the sum

$$\sum_{k=0}^{n}a_kb_{n-k}=\sum_{k=0}^{n}(-1)^{2n}\binom{-1/2}{n}\binom{-1/2}{n-k}e^{(2k-n)i\theta}.$$

Thus we obtain

$$(1-2t\cos\theta+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{-1/2}{n}\binom{-1/2}{n-k}e^{(2k-n)i\theta}.$$

Comparing the real coefficients next to t^n from the previous and the initial equation, for $x = \cos\theta$ we obtain

$$P_n(\cos\theta) = \text{Re}\left[\sum_{k=0}^{n}\binom{-1/2}{n}\binom{-1/2}{n-k}e^{(2k-n)i\theta}\right] =$$
$$= \sum_{k=0}^{n}\binom{-1/2}{n}\binom{-1/2}{n-k}\cos(2k-n)\theta. \tag{5.259}$$

This equation can be expressed in the following form

$$P_n(\cos\theta) = \sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\frac{(2n-2k-1)!!}{(2n-2k)!!}\cos(n-2k)\theta.$$

Special forms of (5.259) are used in practice for even values of n

$$P_n(\cos\theta) = \binom{-1/2}{n}^2 + 2\sum_{k=0}^{\frac{n}{2}-1}\binom{-1/2}{n}\binom{-1/2}{n-k}\cos(2k-n)\theta,$$

that is, for odd values of n

$$P_n(\cos\theta) = -2\sum_{k=0}^{\frac{n-1}{2}}\binom{-1/2}{n}\binom{-1/2}{n-k}\cos(2k-n)\theta.$$

Laguerre polynomials

Problem 198

Prove that Laguerre polynomials are solutions of Laguerre differential equation

$$xy'' + (1-x)y' + ny = 0.$$

Solution

Laguerre polynomials are generated by the function

$$G(x,t) = \frac{1}{1-t}e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty}L_n(x)\frac{t^n}{n!}. \tag{5.260}$$

If both sides of the equation are differentiated by t, and then multiplied by $1-t^2$, we obtain

$$e^{-\frac{xt}{1-t}} - \frac{x}{1-t}e^{-\frac{xt}{1-t}} = (1-t^2)\sum_{n=1}^{\infty}L_n(x)\frac{t^{n-1}}{(n-1)!}.$$

By applying (5.260) we further obtain

$$(1-t)\sum_{n=0}^{\infty}L_n(x)\frac{t^n}{n!} - x\sum_{n=0}^{\infty}L_n(x)\frac{t^n}{n!} = (1-t^2)\sum_{n=1}^{\infty}L_n(x)\frac{t^{n-1}}{(n-1)!},$$

and from there, equating coefficients next to t^n, the recurrent relation

$$L_{n+1}(x)+(x-2n-1)L_n(x)+n^2L_{n-1}(x)=0. \tag{5.261}$$

If the equation (5.260) is now differentiated by x we obtain

$$-\frac{t}{(1-t)^2}e^{-\frac{xt}{1-t}}=\sum_{n=0}^{\infty}L_n'(x)\frac{t^n}{n!}.$$

Using (5.260) we further obtain

$$t\sum_{n=0}^{\infty}L_n(x)\frac{t^n}{n!}+(1-t)\sum_{n=0}^{\infty}L_n'(x)\frac{t^n}{n!}=0.$$

Equating coefficients next to t^n yields the recurrent formula

$$nL_{n-1}(x)+L_n'(x)-nL_{n-1}'(x)=0. \tag{5.262}$$

If the equation (5.261) is differentiated twice by x, we obtain

$$L_{n+1}''(x)+(x-2n-1)L_n''(x)+2L_n'(x)+n^2L_{n-1}''(x)=0.$$

Further, substituting n by $n+1$, yields

$$L_{n+2}''(x)+(x-2n-3)L_{n+1}''(x)+2L_{n+1}'+(n+1)^2L_n''(x)=0. \tag{5.263}$$

From (5.262) it follows that

$$L_n'(x)=nL_{n-1}'(x)-nL_{n-1}(x),$$

and then differentiating by x

$$L_n''(x)=nL_{n-1}''(x)-nL_{n-1}'(x).$$

If we now substitute n by $n+1$ and $n+2$ respectively, we obtain

$$L_{n+1}'(x)=(n+1)L_n'(x)-(n+1)L_n(x) \tag{5.264}$$
$$L_{n+1}''(x)=(n+1)L_n''(x)-(n+1)L_n'(x), \tag{5.265}$$
$$L_{n+2}''(x)=(n+2)L_{n+1}''(x)-(n+2)L_{n+1}'(x). \tag{5.266}$$

Substituting $L_{n+1}'(x)$ from (5.264) and $L_{n+1}''(x)$ from (5.265) into (5.266), yields

$$\begin{aligned}L_{n+2}''(x)&=(n+2)\{(n+1)[L_n''(x)-L_n'(x)]-(n+1)[L_n'(x)-L_n(x)]\}=\\&=(n+1)(n+2)L_n''(x)-2(n+1)(n+2)L_n'(x)+(n+1)(n+2)L_n(x).\end{aligned}$$

Using the obtained results, the equation (5.263) can be reduced to the following form

$$xL_n''(x)+(1-x)L_n'(x)+nL_n(x)=0,$$

and we thus conclude that the Laguerre polynomial $L_n(x)$ is a particular solution of the Laguerre differential equation.

Hermite polynomials

Problem 199

Prove the recurrence formula

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x),$$

bearing in mind that the function generating Hermite polynomials $H_n(x)$ has the following form

$$G(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Solution

Let us start from the equation

$$G(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Differentiating by t yields

$$(2x-2t)e^{2xt-t^2} = \sum_{n=1}^{\infty} H_n(x)n\frac{t^{n-1}}{n!},$$

and from there it follows that

$$(2x-2t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

$$2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!}$$

$$2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)(n+1)\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!}$$

$$2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty} H_{n-1}(x)n\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!}.$$

Equating coefficients next to t^n yields

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x).$$

Bessel polynomials

Problem 200

Prove

$$\frac{1}{2}(J_{n-1}(z) - J_{n+1}(z)) = J_n'(z),$$

bearing in mind that Bessel polynomials $J_n(z)$ are generated by the function

$$G(z,t) = e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n. \tag{5.267}$$

Solution

Differentiating the equation (5.267) by z yields

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n'(z)t^n,$$

that is

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(z)t^n = \sum_{n=-\infty}^{\infty} J_n'(z)t^n$$

or

$$\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_n(z)t^{n+1} - \sum_{n=-\infty}^{\infty} J_n(z)t^{n-1}\right) = \sum_{n=-\infty}^{\infty} J_n'(z)t^n.$$

All sums are reduced to t^n, and given that n takes values from $-\infty$ to ∞, we obtain

$$\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_{n-1}(z)t^n - \sum_{n=-\infty}^{\infty} J_{n+1}(z)t^n\right) = \sum_{n=-\infty}^{\infty} J_n'(z)t^n.$$

Equating coefficients next to t^n we obtain

$$\frac{1}{2}(J_{n-1}(z) - J_{n+1}(z)) = J_n'(z).$$

Problem 201

Prove the following identities

$$J_{n-1}(z) = \frac{n}{z}J_n(z) + J_n'(z),$$
$$J_{n+1}(z) = \frac{n}{z}J_n(z) - J_n'(z),$$

bearing in mind that the Bessel function can be represented by the series

$$J_n(z) = \sum_{r=0}^{\infty}(-1)^r \frac{(z/2)^{n+2r}}{r!(n+r)!}. \tag{5.268}$$

Solution

Differentiating (5.268) by z we obtain

$$J_n'(z) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)\, z^{n+2r-1}}{2^{n+2r} r!(n+r)!}. \tag{5.269}$$

If equation (5.268) is now multiplied by $\frac{n}{z}$, and then added to equation (5.269), we obtain

$$\frac{n}{z} J_n(z) + J_n'(z) = \sum_{r=0}^{\infty} (-1)^r \frac{(2n+2r)\, z^{n+2r-1}}{2^{n+2r} r!(n+r)!}.$$

The right hand side of this equation is

$$\sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)\, z^{n+2r-1}}{2^{n+2r} r!(n+r)!} = \sum_{r=0}^{\infty} (-1)^r \frac{z^{n+2r-1}}{2^{n+2r-1} r!(n+r-1)!} = J_{n-1}(z),$$

which proves the first identity.

In order to prove the second identity, we will first multiply the equation (5.268) by $\frac{n}{z}$, and then substitute r by $r+1$, which yields

$$\frac{n}{z} J_n(z) = \sum_{r=0}^{\infty} (-1)^r \frac{n z^{n+2r-1}}{2^{n+2r} r!(n+r)!} = \tag{5.270}$$

$$= \sum_{r=-1}^{\infty} (-1)^{r+1} \frac{n z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!}. \tag{5.271}$$

If we substitute r by $r+1$ in equation (5.269), we obtain

$$J_n'(z) = \sum_{r=-1}^{\infty} (-1)^{r+1} \frac{(n+2r+2) z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!}. \tag{5.272}$$

Subtracting equation (5.272) from equation (5.271) yields

$$\frac{n}{z} J_n(z) - J_n'(z) = \sum_{r=-1}^{\infty} (-1)^{r+1} \frac{n z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!} -$$

$$- \sum_{r=-1}^{\infty} (-1)^{r+1} \frac{(n+2r+2) z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!} =$$

$$= \sum_{r=0}^{\infty} (-1)^{r+1} \frac{n z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!} + (-1)^0 \frac{n z^{n-1}}{2^n 0! n!} -$$

$$- \sum_{r=0}^{\infty} (-1)^{r+1} \frac{(n+2r+2) z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!} - (-1)^0 \frac{n z^{n-1}}{2^n 0! n!}$$

$$= \sum_{r=0}^{\infty} (-1)^{r+1} \frac{(-2r-2) z^{n+2r+1}}{2 \cdot 2^{n+2r+1} (r+1)!(n+r+1)!} =$$

$$= \sum_{r=0}^{\infty} (-1)^{r} \frac{z^{n+2r+1}}{2^{n+2r+1} r!(n+r+1)!} = J_{n+1}(z),$$

which was to be proved.

Gram–Schmidt process

Problem 202

Determine the first three terms of Legendre polynomials by the Gram–Schmidt process, if the standard base $\{1, x, x^2\}$ is given.

Solution

Legendre polynomials must fulfill the orthogonality condition

$$\int_{-1}^{1} p(x)q(x)\,\mathrm{d}x = 0.$$

If the first term is $y_1 = 1$, then

$$y_2 = x - \frac{\int_{-1}^{1} x \cdot 1\,\mathrm{d}x}{\int_{-1}^{1} 1 \cdot 1\,\mathrm{d}x} = x,$$

$$y_3 = x_2 - \frac{\int_{-1}^{1} x_2 \cdot 1\,\mathrm{d}x}{\int_{-1}^{1} 1 \cdot 1\,\mathrm{d}x}\,1 - \frac{\int_{-1}^{1} x_2 \cdot 1\,\mathrm{d}x}{\int_{-1}^{1} x \cdot x\,\mathrm{d}x}\,x = x_2 - \frac{1}{3}.$$

Theorem 18

If the Sturm–Liouville problem (5.229), (5.230) satisfies the conditions of the previous theorem, and if $p \neq 0$ in the entire interval $a \leq x \leq b$, then the main values of the problem are real.

Solution

Let us assume the opposite, namely, that $\lambda = \alpha + i\beta$ is a main value of the problem, and the respective main function has the form

$$y(x) = u(x) + iv(x).$$

In these expressions α, β are real constants, and u and v are real functions.

By substituting these values into equation (5.229) we obtain

$$\left(ru' + irv'\right)' + (q + \alpha p + i\beta p)\,(u + iv) = 0.$$

In order for this complex equation to be satisfied, it is necessary that both its real and imaginary parts be equal to zero, i.e.

$$\left(ru'\right)' + (q + \alpha p)\,u - \beta pv = 0, \quad \left(rv'\right)' + (q + \alpha p)\,v + \beta pu = 0.$$

By multiplying the first equation by v, and the second equation by $-u$, and then adding the resulting equations, we obtain

$$-\beta\left(u^2+v^2\right)p = u(rv')' - v(ru')' = \left[(rv')u-(ru')v\right]'.$$

The expression in the square brackets is a continuous function in the interval $a \le x \le b$ (see proof of the previous theorem), and thus by integrating, bearing in mind the boundary conditions (as in the previous theorem), we obtain

$$-\beta\int_a^b \left(u^2+v^2\right)p\,\mathrm{d}x = \left[r\left(uv'-vu'\right)\right]\Big|_a^b = 0.$$

As y is a main function, it follows that $y \not\equiv 0$. Further, as y and p are continuous functions, where $p>0$ or $p<0$ in the interval $a \le x \le b$, and $y^2 = u^2+v^2 \not\equiv 0$, it follows that the integral on the left hand side of the last equation is different from zero. From here, it follows that β must be equal to 0, i.e. $\beta = 0$. Given that $\lambda = \alpha + i\beta$ and $\beta = 0$, it follows that $\lambda = \alpha$, that is, λ is a real number. The theorem is thus proven.

Mittag-Leffler (ML) functions

Problem 203

Prove that $E_1(\pm x)$ is an exponential function.

Solution

The ML function is defined by the expression (5.183)

$$E_\alpha(x) = \sum_{k=0} \frac{x^k}{\Gamma(\alpha k+1)}, \qquad \alpha > 0,$$

so that, for $\alpha = 1$,

$$E_1(\pm x) = \sum_{k=0}^{\infty} \frac{(\pm x)^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{(\pm x)^k}{k!} = e^{\pm x}.$$

We have used here the relation $\Gamma(k+1) = k!$ and the expansion of the exponential function into the Taylor (Maclaurin) series.

Problem 204

Prove that $E_{1,2}(x) = 1 + xE_{1,3}(x)$.

Solution

ML function of two parameters (5.184) is defined by the following series

$$E_{\alpha,\beta}(x) = \sum_{k=0} \frac{x^k}{\Gamma(\alpha k+\beta)}.$$

In our case, $\alpha = 1$, $\beta = 2$, and thus

$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{x^0}{\Gamma(2)} + \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(k+2)}.$$

Given that $x^0 = 1$ and $\Gamma(2) = 1! = 1$, it follows that

$$E_{1,2}(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(k+2)}.$$

If we introduce the substitution $k = n+1$ we obtain

$$E_{1,2}(x) = 1 + \sum_{n+1=1}^{\infty} \frac{x^{n+1}}{\Gamma(n+3)} = 1 + \sum_{n+1=1}^{\infty} \frac{x \cdot x^n}{\Gamma(n+3)} =$$
$$= 1 + x \underbrace{\sum_{n+1=1}^{\infty} \frac{x^n}{\Gamma(n+3)}}_{E_{1,3}}$$

which was to be proved.

Problem 205

Find $E_0(x)$, if $|x| < 1$.

Solution

The ML function, for $\alpha = 0$, is

$$E_0(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(0+1)} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots.$$

Given that $\Gamma(0+1) = 1$, this series, for $|x| < 1$, becomes the so called geometric series, the sum of which is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Thus

$$E_0(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Problem 206

Determine the first derivative of the function $E_\alpha(x^\alpha)$, for $\alpha > 0$ and $x > 0$, and then find its value for $\alpha = 1$.

Solution

The derivative of the ML function is

$$\frac{\mathrm{d}}{\mathrm{d}x}E_\alpha(x^\alpha) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(\alpha k+1)} = \frac{\mathrm{d}}{\mathrm{d}x}\left[1+\sum_{k=1}^{\infty}\frac{x^{\alpha k}}{\Gamma(\alpha k+1)}\right] = \sum_{k=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\alpha k}}{\Gamma(\alpha k+1)}\right) =$$
$$= \sum_{k=0}^{\infty}\frac{(\alpha k)x^{\alpha k-1}}{\Gamma(\alpha k+1)}.$$

As $\Gamma(z+1) = z\Gamma(z)$, i.e. in our case $\Gamma(\alpha k+1) = (\alpha k)\Gamma(\alpha k)$, it follows that the derivative of the ML function is

$$\sum_{k=0}^{\infty}\frac{(\alpha k)x^{\alpha k-1}}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty}\frac{\cancel{(\alpha k)}x^{\alpha k-1}}{\cancel{(\alpha k)}\Gamma(\alpha k)},$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}x}E_\alpha(x^\alpha) = \sum_{k=1}^{\infty}\frac{x^{\alpha k-1}}{\Gamma(\alpha k)} = \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^{\alpha k}}{\Gamma(\alpha k)}.$$

By substituting $k = n+1$, we obtain

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{\alpha(n+1)}}{\Gamma(\alpha(n+1))} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{\alpha}\cdot x^{\alpha n}}{\Gamma(\alpha n+\alpha)} =$$
$$= x^{\alpha-1}\sum_{n=0}^{\infty}\frac{x^{\alpha n}}{\Gamma(\alpha n+\alpha)} = x^{\alpha-1}E_{\alpha,\alpha}(x^\alpha).$$

Thus, this derivative is given by the expression

$$\frac{\mathrm{d}}{\mathrm{d}x}E_\alpha(x^\alpha) = x^{\alpha-1}E_{\alpha,\alpha}(x^\alpha).$$

In the special case, when $\alpha = 1$, we obtain

$$\frac{\mathrm{d}E_1(x)}{\mathrm{d}x} = E_{1,1}(x) = e^x,$$

which was to be expected, because $E_1(x) = e^x$, and the derivative of this function is the function itself.

R Note that we have used here a property of convergent series (see Properties of uniformly convergent series, p. 225, eq. (5.7)).

Problem 207

Let $k \in \mathbb{N}$ and $E_k(\cdot)$ be the ML function. Prove that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k E_k(x^k) = E_k(x^k).$$

Solution

The k-th derivative of the ML function can be calculated as follows

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{(k)} E_k(x^k) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\sum_{l=0}^{\infty}\frac{x^{kl}}{\Gamma(kl+1)}\right] = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[1+\sum_{l=1}^{\infty}\frac{x^{kl}}{\Gamma(kl+1)}\right] =$$
$$= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum_{l=1}^{\infty}\frac{x^{kl}}{\Gamma(kl+1)}\right) = \sum_{l=1}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{x^{kl}}{\Gamma(kl+1)}\right) =$$
$$= \sum_{l=1}^{\infty}\frac{1}{\Gamma(kl+1)}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^{kl}\right),$$

given that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^{kl}\right) = (kl)(kl-1)\cdots(kl-k+1)\cdot x^{kl-k} =$$
$$= \frac{(kl)(kl-1)\cdots(kl-k+1)\cdot\overbrace{(kl-k)(kl-k-1)\cdot 2\cdot 1}^{(kl-k)!}\cdot x^{kl-k}}{(kl-k)!} =$$
$$= \frac{(kl)!}{(kl-k)!}x^{kl-k} = \frac{\Gamma(kl+1)}{\Gamma(kl-k+1)}x^{kl-k}.$$

The following relations were also used

$$(kl)! = \Gamma(kl+1), \quad (kl-k)! = \Gamma(kl-k+1).$$

Thus, the k-th derivative can be expressed as

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{(k)} E_k(x^k) = \sum_{l=1}^{\infty}\frac{1}{\cancel{\Gamma(kl+1)}}\frac{\cancel{\Gamma(kl+1)}}{\Gamma(kl-k+1)}x^{kl-k} = \sum_{l=1}^{\infty}\frac{x^{kl-k}}{\Gamma(kl-k+1)}.$$

Introducing the substitution $l = n+1$, we finally obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{(k)} E_k(x^k) = \sum_{n=0}^{\infty}\frac{x^{kn}}{\Gamma(kn+1)} = E_k(x^k).$$

Problem 208

Let $x \in \mathbb{R}$. Determine the value $x E_{2,2}(-x^2)$.

Solution

Given that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

it follows that

$$E_{2,2}(-x^2) = \sum_{k=0}^{\infty}\frac{(-x^2)^k}{\Gamma(2k+2)},$$

and thus

$$xE_{2,2}(-x^2) = x\sum_{k=0}^{\infty}\frac{(-x^2)^k}{\Gamma(2k+2)}.$$

Given that $\Gamma(2k+2) = (2k+1)!$ the previous equation can be expressed as

$$xE_{2,2}(-x^2) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

The series on the right hand side is the Maclaurin series for the sinus function

$$\sin x = \sum_{\ell=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

and thus

$$xE_{2,2}(-x^2) = \sin x.$$

Problem 209

Let $\alpha > 0$ i $x \in \mathbb{R}$. Prove the validity of the so called duplication formula

$$\frac{1}{2}\left[E_\alpha(\sqrt{x}) + E_\alpha(-\sqrt{x}) = E_{2\alpha}(x)\right] = E_{2\alpha}(x).$$

Solution

According to the definition

$$E_\alpha(\pm\sqrt{x}) = \sum_{k=0}^{\infty}\frac{(\pm\sqrt{x})^k}{\Gamma(\alpha k+1)},$$

and the required sum is thus

$$E_\alpha(\sqrt{x}) + E_\alpha(-\sqrt{x}) = \sum_{k=0}^{\infty}\frac{(\sqrt{x})^k + (-1)^k(\sqrt{x})^k}{\Gamma(2\alpha k+1)} = \sum_{k=0}^{\infty}\frac{(\sqrt{x})^k(1+(-1)^k)}{\Gamma(2\alpha k+1)}.$$

Given that

$$(-1)^k = \begin{cases} 1, & \text{if } k = 2n, \\ 0, & \text{if } k = 2n+1, \end{cases}$$

it follows that

$$E_\alpha(\sqrt{x}) + E_\alpha(-\sqrt{x}) = E_{2\alpha}(x) = 2\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(2\alpha n+1)} = E_{2\alpha}(x).$$

Problem 210

Let $|x| < 1$ and $\alpha > 0$. Prove the validity of the following equation

$$\int\limits_0^\infty e^{-t}\, t^{\beta-1} E_{\alpha,\beta}(xt^\alpha)\, \mathrm{d}t = \int\limits_0^\infty e^{-t} E_\alpha(xt^\alpha)\, \mathrm{d}t = \frac{1}{1-x}$$

Solution

According to the definition of the ML function with two parameters

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},$$

that is, in our case

$$E_{\alpha,\beta}(xt^\alpha) = \sum_{k=0}^{\infty} \frac{(xt^\alpha)^k}{\Gamma(\alpha k+\beta)}$$

the required integral is

$$\int_0^\infty e^{-t}\, t^{\beta-1} \sum_{k=0}^{\infty} \frac{(xt^\alpha)^k}{\Gamma(\alpha k+\beta)}\, \mathrm{d}t.$$

The same as in the case of derivation, we can alter here the sequence of the operations of integration and addition (convergent series!), and thus obtain

$$\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \int_0^\infty e^{-t}\, t^{\beta-1}\, t^{\alpha k}\, \mathrm{d}t = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \underbrace{\int_0^\infty e^{-t}\, t^{\alpha k+\beta-1}\, \mathrm{d}t}_{\Gamma(\alpha k+\beta)} =$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \cdot \Gamma(\alpha k+\beta) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

(for $|x| < 1$), and it further follows that

$$\int\limits_0^\infty e^{-t}\, t^{\beta-1} E_{\alpha,\beta}(xt^\alpha)\, \mathrm{d}t = \frac{1}{1-x}.$$

The right hand side of the equation does not depend on parameters α and β, and thus, if for the integral on the left hand side $\beta = 1$ is chosen, we obtain

$$\int\limits_0^\infty e^{-t} E_\alpha(xt^\alpha)\, \mathrm{d}t = \frac{1}{1-x},$$

which was also to be proved.

Problem 211

Let $\alpha > 0$, $\beta > 0$, $\rho > 0$, $x > 0$ and $a \in \mathbb{R}$. Prove that

$$I \equiv \int_0^x t^{\beta-1} E^{\rho}_{\alpha,\beta}(at^{\alpha})\,dt = x^{\beta} E^{\rho}_{\alpha,\beta}(at^{\alpha}),$$

where $E^{\rho}_{\alpha,\beta}(at^{\alpha})$ is the ML function with three parameters.

Solution

Using the definition of the ML function with three parameters, for the given integral we obtain

$$I = \int_0^x t^{\beta-1} E^{\rho}_{\alpha,\beta}(at^{\alpha})\,dt = \int_0^x t^{\beta-1} \sum_{k=0}^{\infty} \frac{(\rho_k}{\Gamma(\alpha k+\beta)} \frac{(at^{\alpha})^k}{k!}\,dt =$$

$$= \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)} \frac{a^k}{k!} \int_0^x t^{\beta-1}\cdot t^{\alpha k}\,dt = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)} \frac{a^k}{k!} \int_0^x t^{\alpha k+\beta-1}\,dt =$$

$$= \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)} \frac{a^k}{k!} \frac{x^{\alpha k+\beta}}{\alpha k+\beta} = \left\{ \sum_{k=0}^{\infty} \frac{(\rho)_k}{\underbrace{(\alpha k+\beta)\Gamma(\alpha k+\beta)}_{\Gamma(\alpha k+\beta+1)}} \cdot \frac{(ax^{\alpha})^k}{k!} \right\} \cdot x^{\beta}.$$

The expression in brackets is the ML function with three parameters, and thus

$$I = x^{\beta} E^{\rho}_{\alpha,\beta}(ax^{\alpha}),$$

which was to be proved.

Problem 212

Compute the value of the integral $I = \int_0^x t^{\beta-1} E^{\rho}_{\alpha,\beta}(at^{\alpha})\,dt$ (from Example 211), for parameter values: $a = -1$, $\alpha = 2$ i $\beta = 1 = \rho$.

Solution

Note, that in this special case, $E^1_{2,1}(\cdot) = E_2(\cdot)$. Namely, as by definition

$$E^{\rho}_{\alpha,\beta}(\cdot) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)} \cdot \frac{(\cdot)^k}{k!} = \sum_{k=0}^{\infty} \frac{\Gamma(\rho+k)}{\Gamma(\rho)} \frac{1}{\Gamma(\alpha k+\beta)} \frac{(\cdot)^k}{k!} \quad \Rightarrow$$

$$E^1_{\alpha,1}(\cdot) = \sum_{k=0}^{\infty} \frac{\cancel{\Gamma(1+k)}}{\Gamma(1)} \frac{1}{\Gamma(\alpha k+1)} \frac{(\cdot)^k}{\cancel{k!}} = \sum_{k=0}^{\infty} \frac{(\cdot)^k}{\Gamma(\alpha k+1)} = E_{\alpha}(\cdot),$$

it follows that

$$E^1_{2,1}(\cdot) = E_2(\cdot),$$

and thus

$$I = \int_0^x t^0 E_{2,1}^1(-t^\alpha)\,dt = \int_0^x E_2(-t^\alpha)\,dt.$$

Further, given that

$$E_2(-t^2) = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k (t^2)^k}{(2k)!} = \cos t$$

the integral I is equal to

$$I = \int_0^x E_2(-t^2)\,dt = \int_0^x \cos t\,dt = \sin t.$$

Using the result from Example 208, we obtain

$$\int_0^x E_2(-t^\alpha)\,dt = x E_{2,2}(-x^2)$$

which is the required result.

Problem 213

Let $\alpha > 0$. Prove that

$$E_\alpha(-x) = E_{2\alpha}(x^2) - x E_{2\alpha,\alpha+1}(x^2)$$

is valid for ML functions.

Solution

Using the definition for ML functions with one and with two parameters, the right hand side becomes

$$\Omega \equiv E_{2\alpha}(x^2) - x E_{2\alpha,\alpha+1}(x^2) = \sum_{k=0}^{\infty} \frac{(x^2)^k}{\Gamma(2\alpha k+1)} - x \sum_{k=0}^{\infty} \frac{(x^2)^k}{\Gamma(2\alpha k\alpha+1)},$$

or in the expanded form

$$\Omega = \left\{1 + \frac{x^2}{\Gamma(2\alpha+1)} + \frac{x^4}{\Gamma(4\alpha+1)} + \frac{x^6}{\Gamma(6\alpha+1)} + \cdots\right\} - \\ - \left\{\frac{x}{\Gamma(\alpha+1)} + \frac{x^3}{\Gamma(3\alpha+1)} + \frac{x^5}{\Gamma(5\alpha+1)} + \cdots\right\}.$$

Grouping the terms next to each power of $-x$, we obtain

$$\Omega = \left\{\frac{(-x)}{\Gamma(\alpha+1)} + \frac{(-x)^2}{\Gamma(2\alpha+1)} + \frac{(-x)^3}{\Gamma(3\alpha+1)} + \frac{(-x)^4}{\Gamma(4\alpha+1)} + \frac{(-x)^5}{\Gamma(5\alpha+1)} + \cdots\right\} = \\ = \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(k\alpha+1)} = E_\alpha(-x).$$

Thus

$$E_{2\alpha}(x^2) - xE_{2\alpha,\alpha+1}(x^2) = E_\alpha(-x).$$

which was to be proved.

Elliptic functions

Problem 214

Determine the final equations of motion of a heavy point M, of mass m, along a smooth circular vertical fixed line - the mathematical pendulum.

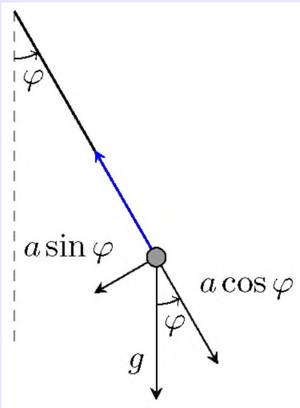

Figure 5.11: Mathematical pendulum.

Solution

Motion of a heavy point along a vertical fixed circular line, without friction, is called the **mathematical pendulum** (Fig. 5.11).

The pendulum moves in a vertical plane, say $x-z$, with downward direction of the z-axis (direction of action of gravitational force).

In this case it is convenient to use the polar coordinate system, where the coordinate origin is set at the centre of the circle.

Note first that the position of the point is determined by the angle φ, because the point is always at the same distance R from the coordinate origin.

In order to solve this problem, we will use the law of change of kinetic energy $\mathrm{d}A = \mathrm{d}E_k$, where $\mathrm{d}A = \sum_i \mathbf{F}_i \,\mathrm{d}\mathbf{r}_i$ is the work element, $\mathbf{F}$ is the force, and $E_k = \frac{1}{2}mv^2$ the kinetic energy. In our case, two forces act on the point: gravity $\mathbf{G} = mg\mathbf{k}$ and the reaction resulting from $\mathbf{R} = R_n\mathbf{n}$ (smooth line, so there is no reaction in the direction of the tangent), and thus

$$\mathrm{d}A = \mathbf{G}\cdot\mathbf{r} + \mathbf{R}\cdot\mathbf{r} = mg\mathbf{k}\cdot(\mathrm{d}x\mathbf{i} - \mathrm{d}z\mathbf{k}) + R_n\mathbf{n}\cdot\mathrm{d}r\mathbf{t} = -mg\,\mathrm{d}z \quad \Rightarrow$$

$$A = \int_R^z mg\,\mathrm{d}z = mg(z-R).$$

From the law of change of kinetic energy we obtain

$$\mathrm{d}A = \mathrm{d}E_k \quad \Rightarrow \quad A = E_k - E_{k_0} \quad \Rightarrow \quad \not{m}gz - \not{m}gR = \frac{\not{m}}{2}\left(v^2 - v_0^2\right) \quad \Rightarrow$$
$$\frac{v^2}{2} - gz = \frac{v_0^2}{2} - gR \quad \Rightarrow \quad v^2 - 2gz = v_0^2 - 2gR \equiv 2h,$$

where h is a constant that can be determined from the initial conditions ($t_0 = 0 : v_0 = R\dot{\varphi}_0 \neq 0$):

$$h = \frac{v_0^2}{2} - gz_0 = \frac{v_0^2}{2} - Rg\cos\varphi_0.$$

Thus, we are interested in the equation

$$v^2 = 2gz + 2h \tag{5.273}$$

where $z = R\cos\varphi$, z_0 is the initial height of the pendulum, and $v = R\dot{\varphi}$. From previous equations, we obtain

$$R^{\not{2}}\dot{\varphi}^2 = 2\not{R}g\left(\cos\varphi + \frac{h}{Rg}\right) \quad \Rightarrow \quad \dot{\varphi}^2 = \frac{2g}{R}\underbrace{(\cos\varphi + \frac{h}{Rg})}_{f(\varphi)}. \tag{5.274}$$

As the left hand side is the square of a real value (always non-negative), in order for the point to move, the right hand side must satisfy the condition

$$f(\varphi) = \cos\varphi + \frac{h}{Rg} \geq 0. \tag{5.275}$$

Analysis:

1. The function $f(\varphi)$ has a first order zero, i.e. there exists a zero φ^* of this function, where $\cos\varphi^* = -\frac{h}{Rg}$. At that point the pendulum stops, and in order for it to continue moving, an acceleration different from zero is needed, i.e. $f'(\varphi) \neq 0$

$$f'(\varphi) = -\sin\varphi \neq 0.$$

Thus,

$$\begin{aligned} \cos\varphi^* &= -\frac{h}{Rg}, \\ \sin\varphi^* &= \sqrt{1 - (h/Rg)^2} \neq 0, \end{aligned} \tag{5.276}$$

and the necessary condition is that the expression under the square root is positive, i.e.

$$\left|\frac{h}{Rg}\right| < 1. \tag{5.277}$$

From (5.276) it follows that there exist two positions in which the pendulum stops:

$$\varphi_1^* = -\arccos\left(\frac{h}{Rg}\right) \quad \text{i} \quad \varphi_2^* = \arccos\left(\frac{h}{Rg}\right). \tag{5.278}$$

These positions are symmetric with respect to $\varphi = 0$ and $\varphi = \pi$. From the conditions $f(0) = 1 + h/Rg$ and $|h/Rg| < 1$ it is visible that $f(0) > 0$, and the mathematical pendulum is thus moving along the arc $(\varphi_1^*, \varphi_2^*)$ which contains the point $\varphi = 0$. From (5.276) and (5.273) it follows that

$$\cos\varphi^* = -\frac{h}{Rg} = \cos\varphi_0 - \frac{v_0^2}{2Rg}$$

and as $v_0^2/2Rg > 0$, it also follows that $\cos\varphi_0 > \cos\varphi^*$, and thus $|\varphi_0| < |\varphi_0^*|$, so the initial position φ_0 is in the interval $(\varphi_1^*, \varphi_2^*)$.
Such movement of the mathematical pendulum is called **oscillatory movement**, as the point oscillates between positions that correspond to zeros φ_1^* and φ_2^*.

2. The function $f(\varphi)$ has a second order zero, i.e. from the condition $f(\varphi) = 0$, that is

$$\begin{aligned} \cos\varphi^* &= -\frac{h}{Rg}, \\ \sin\varphi^* &= \sqrt{1-(h/Rg)^2} = 0 \quad \Rightarrow \quad \left|\frac{h}{Rg}\right| = 1. \end{aligned} \tag{5.279}$$

The analysis shows that the point reaches a position in which it stops and remains there, because the velocity and the acceleration are equal to zero.

3. In the case when $|h/Rg| > 1$, i.e. there are no real zeroes, and the point does not stop ($v > 0$), there is progressive movement.

Our task is to determine the final equation of the movement, i.e. find $\varphi(t)$. The analysis shows that only the first case is interesting, because that is when the point is performing oscillatory movement (moves to a point where it stops, then changes direction and continues to move...).

Observe the equation (5.274)

$$R^2\dot{\varphi}^2 = 2Rg\left(\cos\varphi + \frac{h}{Rg}\right). \tag{5.280}$$

where $|h/Rg| < 1$.

Let us introduce the substitution

$$\frac{h}{Rg} = -\cos\gamma \tag{5.281}$$

where $0 < \gamma < \pi$ is the **angular amplitude**. From equations (5.280) and (5.281) (separable differential equation) we obtain

$$\sqrt{\frac{R}{g}}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{2(\cos\varphi - \cos\gamma)}. \tag{5.282}$$

This equation can be expressed in the form[20].

$$\sqrt{\frac{R}{g}}\frac{\mathrm{d}(\varphi/2)}{\mathrm{d}t} = \sqrt{\sin^2\frac{\gamma}{2} - \sin^2\frac{\varphi}{2}}, \tag{5.283}$$

that is, using a substitution (introducing a new variable u)

$$\sin\frac{\varphi}{2} = u\sin\frac{\gamma}{2}$$

we obtain

$$\sqrt{\frac{R}{g}}\mathrm{d}t = \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{5.284}$$

We have introduced here yet another substitution $k^2 = \sin^2\frac{\varphi}{2} < 1$.

By integrating equation (5.284) (with initial conditions: $t_0 = 0$, $\varphi_0 = 0$, $u_0 = 0$) we obtain

$$\sqrt{\frac{R}{g}}t = \int_0^u \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{5.285}$$

The integral in this equation is an elliptic integral of the first kind, and thus for u we obtain

$$u = \mathrm{sn}\left(\sqrt{\frac{R}{g}}t\right)$$

which represents the required final solution.

Problem 215

Determine the final equations of the motion of a material point M along the surface of a smooth sphere - the **spherical pendulum**.

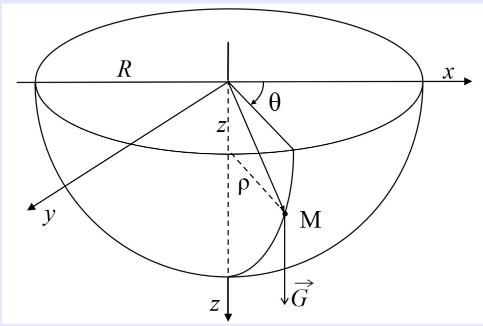

Figure 5.12: Spherical pendulum.

Solution

It is convenient to observe this motion with respect to the spherical coordinate system, in which the equation of the sphere is

$$f(\rho,\theta,z) \equiv \rho^2 + z^2 - R^2 = 0. \tag{5.286}$$

This relation represents the connecting equation!

As in the case of mathematical pendulum, we start from the law on change in kinetic energy

$$A = T - \cancelto{h}{T_0} = \frac{m}{2}v^2 - mh \quad\Rightarrow\quad \frac{1}{2}\cancel{m}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) = \cancel{m}gz + mh$$

[20]Using the relation $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

which yields the differential equation of motion[21]

$$\dot{\rho}^2+\rho^2\dot{\theta}^2+\dot{z}^2=2gz+h, \quad h=\text{const.} \tag{5.287}$$

Thus, three unknowns need to be determined: $\rho=\rho(t)$, $\theta=\theta(t)$, $z=z(t)$. As we have one connecting equation, the degree of freedom is 2 (number of independent coordinates), and we thus need one more equation.

Observe the law on motion momentum

$$\frac{\mathrm{d}\mathbf{L}_0}{\mathrm{d}t}=\mathbf{M}_0,$$

where $\mathbf{L}_0=\mathbf{r}\times m\mathbf{v}$ - is the motion momentum of point O, and $\mathbf{M}_0$ the force momentum of point O.

In this case the following forces are acting: gravity $\mathbf{G}=mg\mathbf{k}$ and the base reaction force $\mathbf{R}$, which has the direction of the normal to the sphere and passes through the coordinate origin. As this force passes through the moment point, it follows that $\mathbf{M}_O^{\mathbf{R}}=\mathbf{0}$, and only the gravity moment needs to be calculated. The gravity has the direction of the z-axis, and thus the projection of the moment to the z-axis $M_z$is equal to 0, i.e. $\dfrac{\mathrm{d}L_z}{\mathrm{d}t}=M_x=0 \quad \Rightarrow \quad L_z=\text{const.}=C$.

From

$$\mathbf{L}_0=\mathbf{r}\times m\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ m\dot{x} & m\dot{y} & m\dot{z}\end{vmatrix}=\mathbf{i}(y\cdot m\dot{z}-z\cdot m\dot{y})+\mathbf{j}(z\cdot m\dot{x}-x\cdot m\dot{z})+\mathbf{k}(x\cdot m\dot{y}-y\cdot m\dot{x})$$

with respect to Cartesian coordinates, we obtain for the motion moment, for the z-axis

$$L_z=m(x\dot{y}-y\dot{x})=\text{const}=C.$$

Given that

$$\begin{aligned} x&=\rho\cos\theta &&\Rightarrow \quad \dot{x}=\dot{\rho}\cos\theta-\rho\dot{\theta}\sin\theta,\\ y&=\rho\sin\theta &&\Rightarrow \quad \dot{y}=\dot{\rho}\sin\theta+\rho\dot{\theta}\cos\theta,\end{aligned}$$

with respect to polar coordinates, we obtain

$$L_z=m\left[\rho\cos\theta\left(\dot{\rho}\sin\theta+\rho\dot{\theta}\cos\theta\right)-\rho\sin\theta\left(\dot{\rho}\cos\theta-\rho\dot{\theta}\sin\theta\right)\right]=\text{const}=C \quad \Rightarrow$$
$$L_z=\rho^2\dot{\theta}=C.$$

The task is to find the position of the point $M(\rho,\theta,z)$, i.e. to determine the final equations of motion

$$\rho=\rho(t), \quad \theta=\theta(t), \quad z=z(t)$$

from the system of differential equations

- the law of change in kinetic energy

$$\dot{\rho}^2+\rho^2\dot{\theta}^2+\dot{z}^2=2gz+h \tag{1}$$

- the law on motion moment

$$\dot{\theta}=\frac{C}{\rho^2} \tag{2}$$

- the connecting equation

$$\rho^2+z^2-R^2=0 \quad \Rightarrow \quad \dot{\rho}=-\frac{z\cdot\dot{z}}{\sqrt{R^2-z^2}}. \tag{3}$$

From (2) i (1) we obtain

$$\dot{\rho}^2+\rho^{\cancel{2}}\frac{C^2}{\rho^{\cancel{4}2}}+\dot{z}^2=2gz+h \tag{4}$$

From (3) and (4) it follows that

$$\frac{z^2\dot{z}^2}{R^2-z^2}+\frac{C^2}{R^2-z^2}+\dot{z}^2=2gz+h \quad \Rightarrow \quad \dot{z}^2\left(1+\frac{z^2}{R^2-z^2}\right)+\frac{C^2}{R^2-z^2}=2gz+h \quad \Rightarrow$$

$$R^2\dot{z}^2=\left(R^2-z^2\right)(2gz+h)-C^2=\underbrace{-2gz^3-z^2h+2gRz+hR^2-C^2}_{P(z)}.$$

Analysis. $P(z)$ is a third degree polynomial. As the left hand side is the square of real functions, only solutions for which $P(z)\geq 0$ make sense. This polynomial has three zeroes, where

$$\lim_{z\to\pm\infty}P(z)=\mp\infty, \quad P(\pm R)=-C^2.$$

In the initial moment the point was at position z_0 and had initial velocity $\dot{z}_0$. Observe the interval $-R\leq z_0\leq R$ in which $P(z_0)=\left(R^2-z_0^2\right)(2gz_0+h)-C^2=R^2\dot{z}_0^2>0$. In order for the point to start moving, it is necessary that the initial velocity is $P(z_0)>0$ (see Figure 5.13).

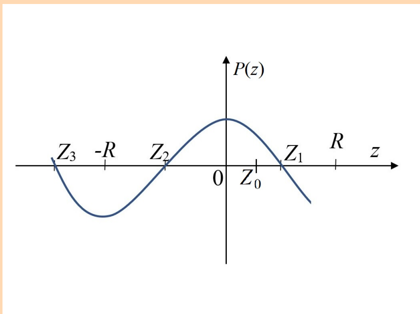

Figure 5.13: Analysis of the task 215 - sketch of the polynomial $P(z)$.

As the polynomial has three zeroes, its sign changes three times (at points: z_1, z_2 i z_3), and we will thus observe the following intervals

$$z_0<z_1<R, \quad -R<z_2<z_0, \quad -\infty<z_3<-R.$$

However, as the point moves along a hemisphere, the z coordinate is limited to the interval $-R\leq z\leq R$, and due to the condition $P(z)>0$, to the interval $z_2\leq z\leq z_1$.

As the polynomial zeroes are z_1, z_2 and z_3, it can be represented in the form

$$P(z)=-g(z-z_1)(z-z_2)(z-z_3).$$

R At the first glance, the interval $-R \le z \le R$ seems impossible, as a hemisphere is in question, and thus $0 < z < R$. However, this would be the case when the point emerges from the hemisphere.

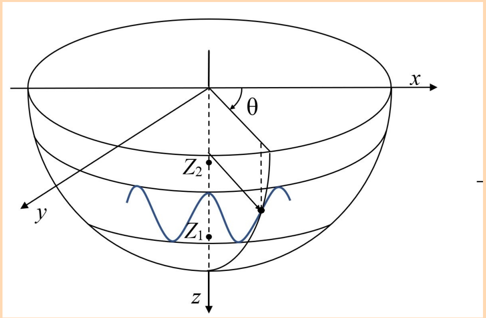

Figure 5.14: The range of possible solutions $-R < z_2 \le z \le z_1 < R$.

Thus, the square of the velocity is determined by the polynomial $P(z)$, and it follows that

$$R^2\dot{z}^2\Big|_{z=z_1,z_2} = P(z_1) = P(z_2) = 0 \quad \Rightarrow \quad \dot{z} = 0 \quad \text{(velocity =0)}.$$

In order for the motion to continue, it is necessary that the acceleration (derivative of motion by time) be different from zero. In our case

$$\begin{aligned}\dot{P}(z) &= -2g\,[(z-z_2)(z-z_3)+(z-z_3)(z-z_1)+(z-z_1)(z-z_2)]\cdot\dot{z} \quad \Rightarrow\\ \dot{P}(z_1) &= -2g\,[(z_1-z_2)(z_1-z_3)+(z_1-z_3)(z_1-z_1)+(z_1-z_1)(z_1-z_2)]\cdot\dot{z} =\\ &= -2g\,\underbrace{(z_1-z_2)}_{>0}\underbrace{(z_1-z_3)}_{>0}\cdot\dot{z} < 0.\end{aligned}$$

The acceleration is different from zero, and it is negative, so the point moves upward (the axis is directed downward, and the acceleration has the opposite sign (direction)!) until the next point where it stops, and that is point z_2. At that moment the acceleration is

$$\begin{aligned}\dot{P}(z_2) &= -2g\,[(z_1-z_2)(z_1-z_3)+(z_1-z_3)(z_1-z_1)+(z_1-z_1)(z_1-z_2)] =\\ &= -2g\,\underbrace{(z_2-z_2)}_{>0}\underbrace{(z_2-z_3)}_{<0} > 0.\end{aligned}$$

and the motion is downward.

Observe now the differential equation of motion, whose solution gives the position of the point with respect to its height (z coordinate).

$$\begin{aligned}&R^2\,\dot{z}^2 = -2g(z-z_1)(z-z_2)(z-z_3), \quad \text{with initial conditions:} \quad z_0 = z_1,\ \dot{z}_0 = 0,\\ &\quad \dot{z} = -2(z_1-z_2)u\cdot\dot{u}\end{aligned} \tag{5.288}$$

and substitution $z = z_1 - (z_1 - z_2)u^2$. The initial value for this variable is

$$z_0 = z_1 = z_1 - (z_1 - z_2)u_0^2 \quad \Rightarrow \quad 0 = (z_1 - z_2)u_0^2 \quad \Rightarrow \quad u_0 = 0.$$

The differential equation (5.288) now has the form

$$R^2\cdot 4(z_1-z_2)^2u^2\dot{u}^2 = 2g(z_1-z_2)u^2\left[(z_1-z_2)(1-u^2)\right]\left[z_1-z_3-(z_1-z_2)u^2\right] \quad \Rightarrow$$
$$4R^2\dot{u}^2 = 2g(1-u^2)\left[z_1-z_3-(z_1-z_2)u^2\right] \quad \Rightarrow$$
$$\dot{u}^2 = \frac{g(z_1-z_2)}{2R^2}\left(1-u^2\right)\left(1-\frac{z_1-z_2}{z_1-z_3}u^2\right)$$

If we introduce the following substitutions

$$\frac{z_1-z_2}{z_1-z_3} = k^2 \quad \text{i} \quad \lambda = \frac{\sqrt{2g(z_1-z_2)}}{2R},$$

we obtain the separable differential equation

$$\dot{u} = \lambda\sqrt{(1-u^2)(1-k^2u^2)} \quad \Rightarrow \quad \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} = \lambda\,\mathrm{d}t.$$

Bearing in mind the initial conditions: $t_0 = 0, u_0 = 0$ we finally obtain

$$I = \int_0^u \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} = \int_0^t \lambda\,\mathrm{d}t = \lambda\,t. \tag{5.289}$$

Thus integral I, as a function of the upper bound, cannot be expressed by a finite number of elementary functions, and it is called the **normal elliptic integral of the first kind**.

R Note that the function u must satisfy the condition $1-u^2 \geq 0$ (the expression under the square root must be non-negative!), that is, $u^2 \leq 1$, and thus this integral can obtain another form by substitution $u = \sin\varphi \quad \Rightarrow \quad \mathrm{d}u = \cos\varphi\,\mathrm{d}\varphi$

$$F(\varphi,k) = \int_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

The constant k is called the **modulus of the elliptic integral**, where $0 \leq k^2 \leq 1$.

It can be proved that, for boundary values $k=0$ and $k=1$, this integral is reduced to elementary functions.

[21]We have used here the expression for velocity in polar coordinates $v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$.

IV

Trigonometric Fourier series. Fourier int

V PDE

6. Trigonometric Fourier series. Fourier integral

6.1 Periodic functions

In the natural sciences and technology, we often encounter processes such as: rotations of individual machine parts (Watt regulator, engine rocker,...), oscillations (clock pendulum ,...). These processes are repeated over time, i.e. they are periodic. Such processes are mathematically described by periodic functions.

Definition

A function $f(x)$ of one variable is **periodic**, if there exists a constant $T \neq 0$ such that

$$f(x+T) = f(x), \qquad \text{for } \forall x. \tag{6.1}$$

The constant T is called the **period** of the function $f(x)$.

Theorem 19

If $f(x)$ is a periodic function with a period T, then its period is also nT, where n is a whole number.

Proof

Using the definition of periodicity, for T we obtain

$$f(x+nT) = f[x+(n-1)T+T] = f[x+(n-1)T] = f[x+(n-2)T+T] = f[x+(n-2)T] = \cdots = f(x+T) = f(x). \tag{6.2}$$

This proves the Theorem.

The smallest constant T ($T > 0$), for which (6.1) is true is called the **fundamental** (primitive) **period** of the function $f(x)$, and nT, a multiple of the fundamental period (non-fundamental period).

R Note 1. It is sufficient to analyze (and draw) a periodic function only on the interval of length T (e.g. from 0 to T). The remaining parts of the function are obtained by moving this part translationally to the left (for $-T$, $-2T$,...) or right (for T, $2T$,...).

R Note 2. If the functions f and g are periodic, with periods T_f and T_g, and if $T_f/T_g = p/q$, then $T^* = qT_f = pT_g$ is a period for both functions, i.e. $f(x) = f(x+T^*)$ and $g(x) = g(x+T^*)$.

The simplest periodic process in physics is described by the function

$$x(t) = A\sin(\omega t + \varphi), \qquad -\infty < t < \infty, \tag{6.3}$$

which is called a **harmonic**. The origin of the name comes from harmonic oscillations, which are described by such functions.

6.1.1 Properties of periodic functions

The properties that follow from the definition of the periodic function are:

a) Inverse function of a periodic function is a multivalued function.

b) Derivative of a periodic function is a periodic function.

c) If a periodic function $f(x)$, with period T, is integrable, then [1]

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x. \tag{6.4}$$

d) A primitive function of a periodic function does not have to be a periodic function.

1

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_a^T f(x)\,\mathrm{d}x + \int_T^{a+T} f(x)\,\mathrm{d}x = \int_a^T f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x.$$

We have used here

$$\int_T^{a+T} f(x)\,\mathrm{d}x = \int_T^{a+T} f(x-T)\,\mathrm{d}x.$$

Substituting $x - T = \bar{x} \quad \Rightarrow \quad \mathrm{d}x = \mathrm{d}\bar{x}$ and bearing in mind that the limits also change: for $x_1 = T$ and $x_2 = a+T$, the corresponding values for the new variable are $\bar{x}_1 = 0$ and $\bar{x}_2 = a$. This finally yields

$$\int_T^{a+T} f(x-T)\,\mathrm{d}x = \int_0^a f(\bar{x})\,\mathrm{d}\bar{x}.$$

6.1.2 Extension of non-periodic functions

Observe an arbitrary non-periodic function $f(x)$, defined on the interval $a \leq x \leq a+T$. Let us now construct a periodic function $F(x)$, with period T, which coincides with the function $f(x)$ on the interval $a \leq x \leq a+T$. The graph of the new function is obtained by moving the function $f(x)$ translationally along the x - axis, to the left or to the right for $\pm T, \pm 2T, \ldots, \pm nT, \ldots$ (see Fig. 6.1).

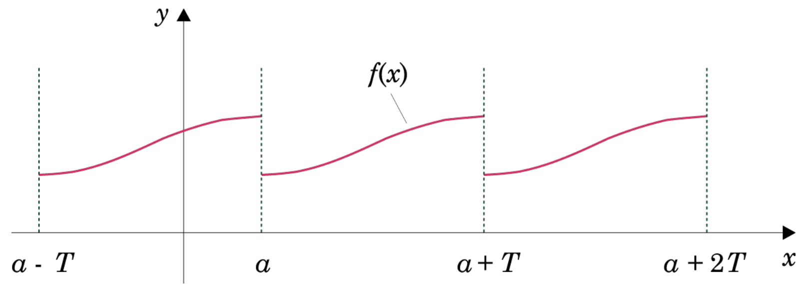

Figure 6.1: Extension of a non-periodic function.

6.1.3 Sum (superposition) of harmonics

Observe the array of harmonics

$$A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right), \qquad k = 1, 2, \ldots, \ -\infty < x < \infty, \ T > 0. \tag{6.5}$$

The period of the k-th harmonic is [2]

$$T_k = \frac{T}{k}. \tag{6.6}$$

The sum (superposition) of a finite number of harmonics is a function of the form

$$f_N(x) = A_0 + \sum_{k=1}^{N} A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right). \tag{6.7}$$

This function is periodic, with period T (see Note 2).

When $N \to \infty$ we obtain an infinite series, namely the function

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right), \tag{6.8}$$

which is also periodic, with period T.

It is well known from trigonometry, that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. Using this relation on the previous series, we obtain

$$f_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N}\left(a_k \cos\frac{k\pi x}{\ell} + b_k \sin\frac{k\pi x}{\ell}\right) \tag{6.9}$$

---

[2]

$$\sin\left[\frac{2\pi k}{T}\left(x + \frac{T}{k}\right) + \varphi_k\right] = \sin\left[\frac{2\pi k}{T}x + \varphi_k + 2\pi\right] = \sin\left(\frac{2\pi k}{T}x + \varphi_k\right),$$

$$\sin\left[\frac{2\pi k}{T}(x+T) + \varphi_k\right] = \sin\left[\frac{2\pi k}{T}x + \varphi_k + \frac{2\pi k}{T}T\right] = \sin\left(\frac{2\pi k}{T}x + \varphi_k + 2\pi k\right) = \sin\left(\frac{2\pi k}{T}x + \varphi_k\right).$$

and

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right) \tag{6.10}$$

We have introduced here the following substitutions

$$\frac{a_0}{2} = A_0, \quad a_k = A_k \sin \varphi_k, \quad b_k = A_k \cos \varphi_k, \quad 2\ell = T.$$

Functions f_N and f defined in this way are periodical with period 2ℓ. The series (6.10) is called a **trigonometric series**. When observing the series (6.10) the following question arises: is it possible to represent a function $f(x)$ by a trigonometric series? The aim of this chapter is to give an answer to this question.

Let us first establish the relation between coefficients a_n and b_n and the function $f(x)$ itself. If the series **converges uniformly** then it can be integrated element-by-element, which yields:

$$\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = a_0\ell + \sum_{k=1}^{\infty} \left[a_k \int_{-\ell}^{\ell} \cos \frac{k\pi x}{\ell} + b_k \int_{-\ell}^{\ell} \sin \frac{k\pi x}{\ell} \right] \Rightarrow$$

$$\Rightarrow \frac{a_0}{2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)\mathrm{d}x. \tag{6.11}$$

Multiplying the relation (6.10) by $\cos \frac{m\pi x}{\ell}$ and $\sin \frac{m\pi x}{\ell}$, respectively, and then integrating from $-\ell$ to ℓ, we obtain:

$$a_m = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{m\pi x}{\ell}\,\mathrm{d}x; \quad b_m = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{m\pi x}{\ell}\,\mathrm{d}x. \tag{6.12}$$

Thus, we have seen how, starting from a trigonometric series, the relation between a function and its corresponding expansion into a series can be established, that is, the relation between the series coefficients and the function.

Coefficients a_k and b_k established in this way are called Euler [3] coefficients of the Fourier [4] series of function $f(x)$.

However, in this way we have not determined whether the Fourier series of the function $f(x)$, with coefficients a_k and b_k determined in this way, converges to the function $f(x)$. Thus, to each integrable function $f(x)$, on the interval $[-\ell, \ell]$, we can relate a trigonometric series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right), \tag{6.13}$$

where coefficients a_k and b_k are determined by expressions (6.12). The symbol " $\sim$ " is used in this case to indicate that the convergence of the observed series has not yet been established. When its convergence (that is, conditions the function $f(x)$ must satisfy) is proved, then the symbol " $\sim$ " can be replaced by "=".

[3] Leonhard Euler (1707-1783), famous Swiss mathematician. He contributed to almost all areas of mathematics and its applications in problems of physics. Special emphasis should be placed on his contribution in the field of differential and difference equations, Fourier series, special functions, complex analysis, calculus of variations, mechanics and hydrodynamics.

[4] Jean-Baptiste Joseph Fourier (1768-1830), French physicist and mathematician. He set the foundations of Fourier series in his principal work *Théorie analytique de la chaleur*.

6.2 The fundamental convergence theorem for Fourier series

Before formulating this theorem, let us first define some terms that are going to be used in this formulation.

Partially smooth functions

Definition

The function $f(x)$ is said to be **partially continuous** on the interval $[a,b]$, if it is continuous in all points of the interval, except for a finite number of points, in which it has first-order discontinuities.

Definition

A partially continuous function $f(x)$, defined on the interval $[a,b]$, is said to be **partially smooth**, if its first derivative $f'(x)$ exists, and this derivative is a continuous function in all points of the interval $[a,b]$, except for a finite number of points, in which the left and right limit values

$$f'(x+0) = \lim_{t\to+0} f'(x+t), \quad f'(x-0) = \lim_{t\to+0} f'(x-t), \tag{6.14}$$

exist. It is also assumed that finite limit values $f'(a+0)$ and $f'(b-0)$ exist in the end points of the interval $[a,b]$.

R Note that besides the term "partially" the term "part-by-part" is also used.

Let us demonstrate a partially smooth function on an example (graphically).

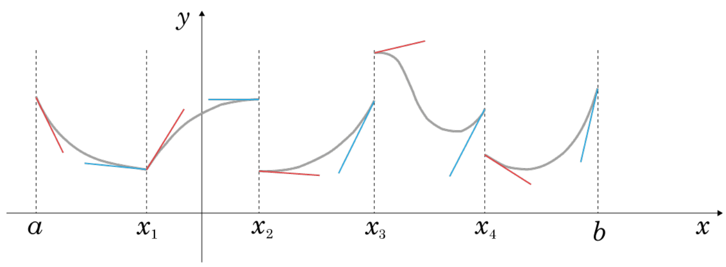

Figure 6.2: Partially smooth function.

Note that this function has tangents at points $x_1,\dots,x_4$, but the tangents on the left and right side are not equal. Thus, the function is smooth in intervals $(a,x_1)\cup(x_1,x_2)\cup(x_2,x_3)\cup(x_3,x_4)$.

The fundamental convergence theorem for Fourier series

Theorem 20

If $f(x)$ is a periodic function, with period 2ℓ, and it is partially smooth on the interval $x \in [-\ell, \ell]$, then the Fourier series of the function $f(x)$ (6.10), the coefficients of which are defined by relations (6.12), is convergent.

The sum of this series

$$s(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right) \tag{6.15}$$

is

a) $s(x_o) = f(x_o)$, if $f(x)$ is continuous at point $x_o \in [-\ell, \ell]$,

b) $s(x_o) = \dfrac{f(x_o+0)+f(x_o-0)}{2}$ at points where the function $f(x)$ has discontinuities.

c) At the ends of the interval the following is true

$$s(-\ell) = s(\ell) = \frac{f(-\ell+0)+f(\ell-0)}{2}.$$

The conditions under which the series converges are known in the literature as Dirichlet [5] conditions.

The proof of this theorem is not difficult, but it requires a lot of space, so we will not elaborate it here. Readers, eager for knowledge, are referred to books [12] and [38].

R Note 1. Given that, for a continuous function, $f(x-0) = f(x+0) = f(x)$, that is

$$\frac{f(x+0)+f(x-0)}{2} = \frac{2f(x)}{2} = f(x),$$

points a) and b) can be simply replaced by

$$s(x) = \frac{f(x+0)+f(x-0)}{2},$$

for all points $x \in [-\ell, \ell]$.

R Note 2. In case it is necessary to expand a function $\varphi(x)$, which is not periodic, but satisfies all other conditions from the previous Theorem, then we proceed as follows. We look for a periodic function $f(x)$, with period 2ℓ, which can be expanded into a Fourier series, and which coincides with the initial function $\varphi(x)$ in the interval $(-\ell, \ell)$, i.e.

$$f(x) = \varphi(x), \quad \text{for} \quad x \in (-\ell, \ell),$$

while outside this interval these functions differ. This can be achieved by extending non-periodic functions, as described earlier.

6.2.1 Expanding even and odd functions into Fourier series. Fourier sine and cosine series

Recall that if the following is true for a function

$$\begin{aligned} f(-x) &= f(x) \quad \text{for} \, \forall x \in [-\ell, \ell] \qquad &\text{the function is even,} \\ f(-x) &= -f(x) \; \text{for} \, \forall x \in [-\ell, \ell] \qquad &\text{the function is odd.} \end{aligned}$$

[5] Peter Gustav Lejeune Dirichlet (1805-1859), German mathematician. He became famous for his important research in the fields of Fourier series and number theory.

Properties of even/odd functions

a) If a function is even, then

$$\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = 2\int_{0}^{\ell} f(x)\,\mathrm{d}x.$$

b) If a function is odd, then

$$\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = 0.$$

c) A product of even functions $p(x) = f(x)\cdot g(x)$ is an even function

$$p(-x) = f(-x)\cdot g(-x) = f(x)\cdot g(x) = p(x).$$

d) A product of odd functions is an even function

$$p(-x) = f(-x)\cdot g(-x) = (-1)f(x)\cdot(-1)g(x) = f(x)\cdot g(x) = p(x).$$

e) A product of an even function $f(x)$ and an odd function $g(x)$ is an odd function

$$p(-x) = f(-x)\cdot g(-x) = f(x)\cdot(-1)g(x) = -f(x)\cdot g(x) = -p(x).$$

These properties are used in calculating the Fourier series coefficients a_k and b_k, when $f(x)$ is an even or odd function. Namely, as the product of $f(x)$ and $\sin x$ ($\sin x$ is an odd function), or $\cos x$ ($\cos x$ is an even function) appear under integral (6.12), then, depending on $f(x)$ and the given the properties a)-d), some coefficients can be equal to zero.

Fourier cosine series and Fourier sine series

Let $f(x)$ be an even function. Then $p(x) = f(x)\cos x$ is an even function, and $q(x) = f(x)\sin x$ is an odd function. Using the properties of these function, we obtain the following Fourier coefficients

$$\begin{aligned}
a_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = \frac{2}{\ell}\int_{0}^{\ell} f(x)\,\mathrm{d}x,\\
a_k &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{k\pi x}{\ell}\,\mathrm{d}x = \frac{2}{\ell}\int_{0}^{\ell} f(x)\cos\frac{k\pi x}{\ell}\,\mathrm{d}x,\\
b_k &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{k\pi x}{\ell}\,\mathrm{d}x = 0.
\end{aligned} \tag{6.16}$$

In this special case, the Fourier series takes the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{\ell}x. \tag{6.17}$$

In this case it is said that the function is expanded into a **Fourier cosine series**, where coefficients a_k are determined by relations (6.16).

Let $f(x)$ be an odd function. Then $p(x) = f(x)\cos x$ is an odd function, and $q(x) = f(x)\sin x$ is an even function. Using the properties of these function, we obtain the following Fourier coefficients

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = 0,$$

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{k\pi x}{\ell}\,\mathrm{d}x = 0,$$

$$b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{k\pi x}{\ell}\,\mathrm{d}x = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{k\pi x}{\ell}\,\mathrm{d}x. \tag{6.18}$$

In this special case, the Fourier series takes the form

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{\ell}x. \tag{6.19}$$

In this case it is said that the function is expanded into a **Fourier sine series**, where coefficients b_k are determined by relations (6.18).

Theorem 21

An arbitrary function $f(x)$, defined on the interval $[-\ell,\ell]$, can be represented as the sum of an even and an odd function in the same interval.

Proof

The function $f(x)$ can be represented in the following form

$$\begin{aligned} f(x) &= f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(-x) = \\ &= \frac{1}{2}f(x) + \frac{1}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(-x) = \\ &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}. \end{aligned}$$

If we introduce the following substitutions

$$f_1(x) = \frac{f(x)+f(-x)}{2} \quad \text{i} \quad f_2(x) = \frac{f(x)-f(-x)}{2},$$

it follows that $f(x) = f_1(x) + f_2(x)$. Let us further prove that the functions f_1 and f_2 are even and odd, respectively.

Given that

$$f_1(-x) = \frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(x)+f(-x)] = f_1(x),$$

$$f_2(-x) = \frac{1}{2}[f(-x) - f(-(-x))] = -\frac{1}{2}[f(x)+f(-x)] = -f_2(x),$$

we can conclude that f_1 is an even, and f_2 an odd function. The theorem is thus proved.

Based on the previous theorem, we conclude that we can always apply the so-called Fourier sine and cosine transformation, but only to parts of a function.

6.2.2 Expansion of functions into Fourier series on the interval $(-\pi,\pi)$

In tis special case $\ell=\pi$, so we obtain that

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx),$$

where the coefficients are determined by the following expressions

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x,\quad a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,\mathrm{d}x,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,\mathrm{d}x.$$

6.2.3 Expansion of functions into Fourier series on the interval $(0,\ell)$. Extension of the half-interval

In various problems of physics and technology, there is a need to expand a function into a Fourier series in a finite interval, $(0,\ell)$. This can be achieved by translating the coordinate system (i.e. introducing a substitution) for $\ell/2$, and then extending, in the general case, a continuous function $f(x)$, as previously explained (choosing ℓ as the period).

Another, more practical way, to do this is the following. Assume that the period is 2ℓ and extend the initial function to the interval $(-\ell,0)$. Since the function is given only on the interval $(0,\ell)$, we can further extend it to the interval $(-\ell,0)$, so that its extension is either an even or an odd function. This procedure is demonstrated on Fig. 6.3.

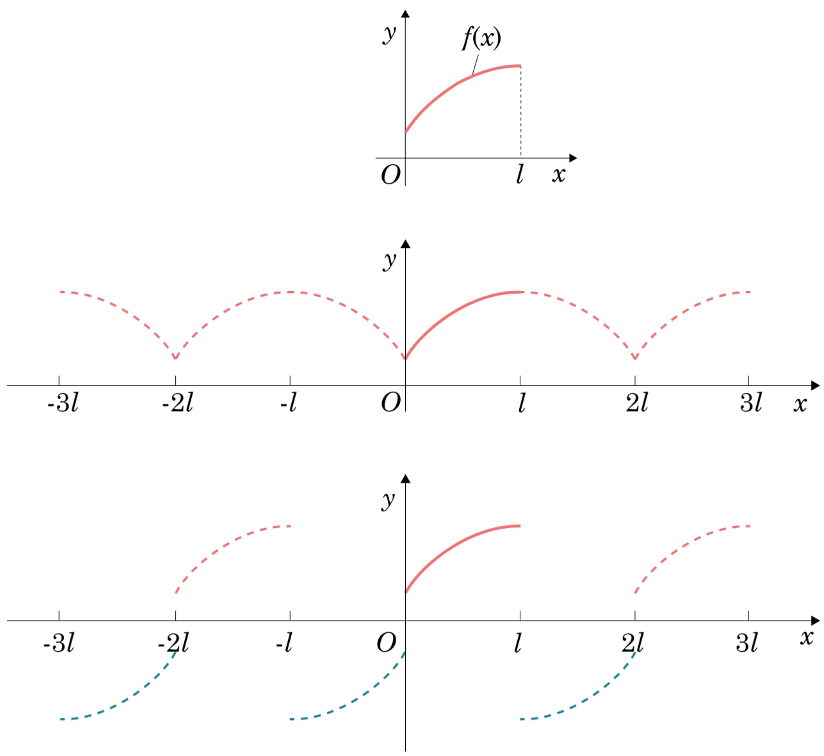

Figure 6.3: Extension of non-periodic function.

This extension is convenient, because in these cases the expansion of a function into a Fourier series is reduced to a Fourier sine or cosine series (due to the parity or oddness of the extended function), and we thus stay in the interval in which the initial function is defined.

Example 216

Expand the following function into a Fourier series

$$f(x)=\begin{cases}\dfrac{2k}{\ell}x, & 0<x<\dfrac{\ell}{2},\\[2mm] \dfrac{2k}{\ell}(\ell-x), & \dfrac{\ell}{2}<x<\ell.\end{cases}$$

Solution

Let us first sketch the given function.

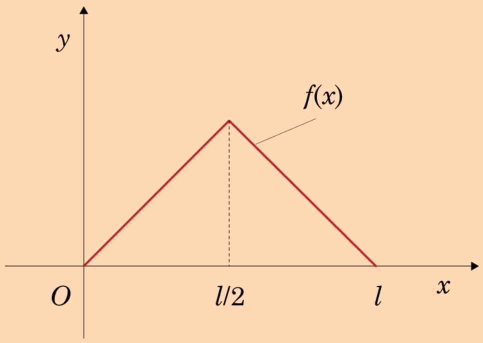

Figure 6.4

According to the conditions of the Fourier series convergence theorem, the function needs to be periodic. Therefore, we will extend the initial function to: a) even and b) odd periodic function.

a) Let us now sketch the extended function, according to the instructions given earlier.

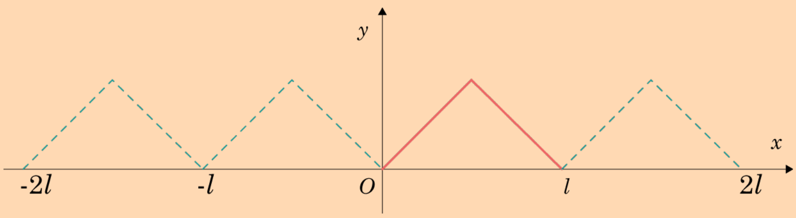

Figure 6.5

In this case (even function), according to (6.16), we obtain

$$\frac{a_0}{2}=\frac{1}{\ell}\left[\frac{2k}{\ell}\int\limits_0^{\ell/2}x\,\mathrm{d}x+\frac{2k}{\ell}\int\limits_{\ell/2}^{\ell}(\ell-x)\,\mathrm{d}x\right],$$

$$a_n=\frac{2}{\ell}\left[\frac{2k}{\ell}\int\limits_0^{\ell/2}x\cos\frac{n\pi}{\ell}x\,\mathrm{d}x+\frac{2k}{\ell}\int\limits_{\ell/2}^{\ell}(\ell-x)\cos\frac{n\pi}{\ell}x\,\mathrm{d}x\right],$$

$$b_n=0.$$

By integrating we obtain the coefficients of the Fourier series

$$\frac{a_0}{2} = \frac{k}{2},$$
$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right),$$
$$b_n = 0.$$

Analyzing the above expressions, we can see that only the terms $n = 2, 6, 10, 14, \ldots$, are different from zero ($a_n \neq 0$), so the Fourier series of the initial even-extended function has the form

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{\ell}x + \frac{1}{6^2}\cos\frac{6\pi}{\ell}x + \ldots\right).$$

b) Let us now sketch the odd extension.

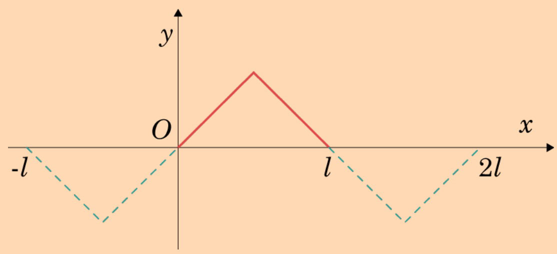

Figure 6.6

In this case (odd function), according to (6.18), we obtain

$$a_n = 0,$$
$$b_n = \frac{2}{\ell}\left[\frac{2k}{\ell}\int_0^{\ell/2} x\sin\frac{n\pi}{\ell}x\,\mathrm{d}x + \frac{2k}{\ell}\int_{\ell/2}^{\ell}(\ell - x)\sin\frac{n\pi}{\ell}x\,\mathrm{d}x\right].$$

From here, by partial integration, we obtain

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Finally, the Fourier series for the odd extension takes the following form

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{\ell}x - \frac{1}{3^2}\sin\frac{3\pi}{\ell}x + \frac{1}{5^2}\sin\frac{5\pi}{\ell}x - \ldots\right).$$

6.2.4 Approximation of a function by a trigonometric polynomial. Mean square error

Let $f(x)$ be a periodic function, with period 2π, which can be represented by a Fourier series.

Let us approximate the observed function by a trigonometric polynomial of N-th order

$$f(x) \approx \frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx) \equiv P_N(x). \tag{6.20}$$

The coefficients of this polynomial, α_k i β_k, are currently indeterminate, and can be expressed in several ways. However, it is clear that we are interested in the form for which the best approximation is obtained, i.e. with the smallest approximation error, for a fixed value of N.

To this end, we first define the approximation error, which can also be done in several ways, but it is most natural to define it by the expression

$$|f(x) - P_N(x)| = \Delta_N, \quad \text{for } x \in [-\ell, \ell].$$

However, in this task, it is more convenient to define the error by an integral, in the form

$$\Delta_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x) - P_N(x)]^2 \, \mathrm{d}x. \tag{6.21}$$

Defined in this way, Δ_N is called the **mean square error**. The task is to determine, for a fixed N, the coefficients α_k i β_k of the polynomial (6.20), so that Δ_N is minimal.

Observe first the subintegral function. Given that $(f - P_N)^2 = f^2 - 2fP_N + P_N^2$, it follows from (6.21) that

$$\Delta_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x) - P_N(x)]^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f^2 - 2fP_N + P_N^2\right] \mathrm{d}x. \tag{6.22}$$

Further, as by the initial assumption, the function $f(x)$ can be represented by a convergent Fourier series, it follows that

$$\int_{-\pi}^{\pi} f\,P_N \, \mathrm{d}x =$$

$$= \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\right] \cdot \left[\frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx)\right] \mathrm{d}x =$$

$$= \pi \left[\frac{a_0 \alpha_0}{2} + \sum_{k=1}^{N} (a_k \alpha_k + b_k \beta_k)\right]. \tag{6.23}$$

We have used here $\int \cos kx \, \mathrm{d}x = 0$, $\int \sin kx \, \mathrm{d}x = 0$ and $\int \sin kx \cos kx \, \mathrm{d}x = 0$.

In a similar way, we also obtain

$$\int_{-\pi}^{\pi} P_N^2 \, \mathrm{d}x = \pi \left[\frac{\alpha_0^2}{2} + \sum_{k=1}^{N} \left(\alpha_k^2 + \beta_k^2\right)\right]. \tag{6.24}$$

The last two relations (6.23) and (6.24) yield

$$\int_{-\pi}^{\pi} \left[P_N^2 - 2fP_N\right] \mathrm{d}x = \pi \left\{\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{N} \left[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2\right]\right\} -$$

$$- \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{N} \left(a_k^2 + b_k^2\right)\right], \tag{6.25}$$

where the following relations have been used

$$(\alpha_k - a_k)^2 = \alpha_k^2 - 2\alpha_k a_k + a_k^2, \qquad \alpha_k^2 - 2a_k\alpha_k = (\alpha_k - a_k)^2 - a_k^2$$
$$(\beta_k - b_k)^2 = \beta_k^2 - 2\beta_k b_k + b_k^2, \qquad \beta_k^2 - 2b_k\beta_k = (\beta_k - b_k)^2 - b_k^2.$$

If we now substitute (6.25) into (6.22), we obatin

$$2\Delta_N = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,\mathrm{d}x - \left[\frac{a_0^2}{2} + \sum_{k=1}^{N}\left(a_k^2 + b_k^2\right)\right] + \tag{6.26}$$
$$+ \left\{\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{N}\left[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2\right]\right\}.$$

As the task is to determine the coefficients α_k and β_k, so that Δ_N is minimal, it can be concluded from (6.26) that the following should be true

$$\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{N}\left[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2\right] = 0. \tag{6.27}$$

It is clear that this term would constantly increase (the sum of squares), with an increase in N, and in this way the error itself would grow. For that reason, we require that it be equal to zero.

From relation (7.70) we obtain the required coefficients

$$\alpha_0 = a_0, \qquad \alpha_k = a_k, \qquad \beta_k = b_k. \tag{6.28}$$

From this, we conclude that the best mean square approximation, for an integrable, periodic function $f(x)$, for $x \in [-\pi, \pi]$, is given by the trigonometric polynomial $P_N(x)$, whose coefficients are the Fourier coefficients of the function $f(x)$.

Some consequences

Note that the error

$$\Delta_N = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - P_N(x)]^2\,\mathrm{d}x$$

is non-negative, as the subintegral function is the square of a real function. Thus, according to (6.26) and (7.70), it follows that

$$2\Delta_N = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x - \left[\frac{a_0^2}{2} + \sum_{k=1}^{N}\left(a_k^2 + b_k^2\right)\right] \geq 0,$$

that is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x \geq \frac{a_0^2}{2} + \sum_{k=1}^{N}\left(a_k^2 + b_k^2\right), \quad \text{za } N = 0, 1, \ldots, \tag{6.29}$$

In literature, this expression (6.29) is known as the **Bessel inequality**. Note now that the left hand side of the inequality is independent of N. It follows that, when $N \to \infty$, the right hand side remains bounded, which means that the series of squares of Fourier coefficients

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

is convergent.

As a consequence of this convergence

$$\lim_{k\to\infty} a_k = 0, \quad \lim_{k\to\infty} b_k = 0,$$

i.e.

$$\lim_{k\to\infty} a_k = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x = 0, \tag{6.30}$$

$$\lim_{k\to\infty} b_k = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\sin kx\,\mathrm{d}x = 0. \tag{6.31}$$

Thus, the Fourier coefficients of a bounded and integrable function tend to zero, when $k \to \infty$.

The relations (6.30) and (6.31) are known in literature as the **Riemann theorem**.

If $\Delta_N \to 0$, when $N \to \infty$, then the inequality (6.29) becomes

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right). \tag{6.32}$$

Let us now demonstrate how the previous relations can be extended to an arbitrary (but finite) period $(-\ell, \ell)$.

Observe a periodic function $\varphi(t)$, with period $(-\pi, \pi)$, i.e. $\varphi(t) = \varphi(t+2\pi)$, and another periodic function $f(x)$, with period $(-\ell, \ell)$, i.e. $f(x) = f(x+2\ell)$. Let us now introduce the substitution $x = \dfrac{\ell}{\pi}t$, where

$$f(x) = f\left(\frac{\ell}{\pi}t\right) = \varphi(t).$$

According to previous substitutions we have

$$\varphi(t+2\pi) = f\left[\frac{\ell}{\pi}(t+2\pi)\right] = f\left(\frac{\ell}{\pi}t + 2\ell\right) = f(x+2\ell) = f(x) = \varphi(t).$$

Further, for trigonometric series, it follows that

$$\varphi(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos kt + b_k\sin kt\right)$$
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{\ell}x + b_k\sin\frac{k\pi}{\ell}x\right),$$

where the appropriate coefficients, say a_k, are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos kt\,\mathrm{d}t = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{k\pi}{\ell}x\,\mathrm{d}x.$$

A substitution of integration boundaries took place, given that

$$\text{for} \quad t_1 = -\pi \quad x_1 = \frac{\ell}{\pi}(-\pi) = -\ell$$
$$\text{and for} \quad t_2 = \pi \quad x_2 = \frac{\ell}{\pi}(\pi) = \ell,$$
$$\text{while the differential is} \quad \mathrm{d}t = \frac{\pi}{\ell}\mathrm{d}x, \quad \text{that is,} \quad \frac{1}{\pi}\mathrm{d}t = \frac{1}{\ell}\mathrm{d}x.$$

We have thus shown how coefficients a_k are computed for an arbitrary period ℓ. In a similar way, the expression for b_k can be obtained.

It can be easily proved that the relation (6.32) also stands for an arbitrary period 2ℓ, i.e.

$$\frac{1}{\ell}\int_{-\ell}^{\ell} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right). \tag{6.33}$$

This relation (6.33) is known as Parseval's [6] identity for the Fourier series.

R Note, that if the function is periodic in interval (a,b), then, as in the previous case, we obtain (for period $b-a$)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{2\pi k}{b-a}(x-a) + b_k \sin\frac{2\pi k}{b-a}(x-a),$$

$$a_k = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2\pi k}{b-a}(x-a)\,\mathrm{d}x,$$

$$b_k = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2\pi k}{b-a}(x-a)\,\mathrm{d}x.$$

6.2.5 Complex form of Fourier series

Using Euler formulas for complex numbers

$$\cos\frac{k\pi x}{\ell} = \frac{e^{\frac{k\pi x}{\ell}i} + e^{-\frac{k\pi x}{\ell}i}}{2}, \qquad \sin\frac{k\pi x}{\ell} = \frac{e^{\frac{k\pi x}{\ell}i} - e^{-\frac{k\pi x}{\ell}i}}{2i},$$

we can represent the Fourier series of the function $f(x)$ in the form

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty}\frac{a_k - b_k i}{2}e^{\frac{k\pi x}{\ell}i} + \sum_{k=1}^{\infty}\frac{a_k + b_k i}{2}e^{-\frac{k\pi x}{\ell}i} = \\ &= \frac{a_0}{2} + \sum_{-\infty}^{-1}\frac{a_k + b_k i}{2}e^{\frac{k\pi x}{\ell}i} + \sum_{1}^{\infty}\frac{a_k - b_k i}{2}e^{\frac{k\pi x}{\ell}i} = \\ &= \sum_{-\infty}^{\infty} c_k e^{\frac{k\pi x}{\ell}i}. \end{aligned}$$

Here, we have

$$c_k = \begin{cases} \dfrac{a_k - b_k i}{2} & k > 0, \\ \dfrac{a_0}{2} & k = 0, \\ \dfrac{a_k + b_k i}{2} & k < 0, \end{cases}$$

where c_k can be determined by the relation

$$c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{\frac{-k\pi x}{\ell}i}\,\mathrm{d}x.$$

[6] Marie Antoine Parseval (1755-1836), famous French mathematician.

6.2.6 Fourier integral

Representation of a function by a Fourier series is widely used in many problems of mathematical physics, but it applies only to periodic functions. We have shown how a function, defined on a finite interval (a,b), can be extended to obtain an even or odd periodic unction. However, functions defined in the interval $(-\infty,\infty)$, which are not periodic, appear in numerous problems. This class of functions cannot be extended into periodic functions in the way described. The question arises: is it possible to extend Fourier's idea to such functions?

Observe some function $f(x)$, which is partly smooth in the interval $[-\ell,\ell]$. Its Fourier series (6.10) has the form

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{\ell}x+b_k\sin\frac{k\pi}{\ell}x\right), \tag{6.34}$$

where coefficients a_k and b_k are determined by relations (6.12)

$$a_k=\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\cos\frac{k\pi}{\ell}t\,\mathrm{d}t,\quad b_k=\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\sin\frac{k\pi}{\ell}t\,\mathrm{d}t,\ k=0,1,2,\ldots \tag{6.35}$$

Substituting (6.35) into (6.34), and given that

$$\cos\frac{k\pi}{\ell}x\cos\frac{k\pi}{\ell}t+\sin\frac{k\pi}{\ell}x\sin\frac{k\pi}{\ell}t=\cos\frac{k\pi}{\ell}(x-t),$$

we obtain

$$f(x)=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(t)\,\mathrm{d}t+\sum_{k=1}^{\infty}\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\cos\frac{k\pi}{\ell}(t-x)\,\mathrm{d}t. \tag{6.36}$$

The previously posed question boils down to the question: what do these integrals tend to when $\ell\to\infty$?

Let us assume that the function $f(x)$ is absolutely integrable in the interval $(-\ell,\ell)$, i.e.

$$\int_{-\ell}^{\ell}|f(t)|\,\mathrm{d}t<M,\quad \text{where } M \text{ is a finite number.}$$

Using this condition, we obtain

$$\lim_{\ell\to\infty}\left|\frac{1}{2\ell}\int_{-\ell}^{\ell}f(t)\,\mathrm{d}t\right|\le\lim_{\ell\to\infty}\frac{1}{2\ell}\int_{-\ell}^{\ell}|f(t)|\,\mathrm{d}t<\lim_{\ell\to\infty}\frac{1}{2\ell}M=0. \tag{6.37}$$

Using (7.297), the relation (6.36) becomes

$$f(x)=\lim_{\ell\to\infty}\sum_{k=1}^{\infty}\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\cos\frac{k\pi}{\ell}(t-x)\,\mathrm{d}t. \tag{6.38}$$

Let us introduce the substitution

$$\alpha_k=\frac{k\pi}{\ell},\quad k=0,1,2,\ldots$$

where

$$\Delta\alpha_k=\alpha_{k+1}-\alpha_k=\frac{(k+1)\pi}{\ell}-\frac{k\pi}{\ell}=\frac{\pi}{\ell} \tag{6.39}$$

and

$$\frac{1}{\ell} = \frac{\Delta\alpha_k}{\pi}. \tag{6.40}$$

From the substitution, it can be seen that the newly introduced variable α_k takes values from the interval $(0, +\infty)$.

Following these substitutions, the relation (6.38) becomes

$$f(x) = \frac{1}{\pi} \lim_{\ell\to\infty} \sum_{k=1}^{\infty} \Delta\alpha_k \int_{-\ell}^{\ell} f(t) \cos\alpha_k(t-x)\,\mathrm{d}t, \tag{6.41}$$

that is, when we switch to the limit value

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} f(t) \cos\alpha(t-x)\,\mathrm{d}t. \tag{6.42}$$

The relation (6.42) is known in literature as the **Fourier formula**, and the corresponding integral as the **Fourier integral**.

Theorem 22

Let the function $f(x)$ be:

- partly smooth in every finite interval, and
- absolutely integrable in interval $(-\infty, \infty)$.

Then the function $f(x)$ can be substituted by a Fourier integral (6.42) for each x, except in points of first order discontinuity x_o, in which the value of the function $f(x_o)$ should be substituted by

$$\frac{f(x_o-0)+f(x_o+0)}{2}.$$

The Fourier formula can also be represented by the relation

$$f(x) = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,\mathrm{d}\lambda, \tag{6.43}$$

where

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,\mathrm{d}x,$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,\mathrm{d}x.$$

6.3 Examples

Problem 217

Expand the function $f(x)$ shown in the figure into a Fourier series

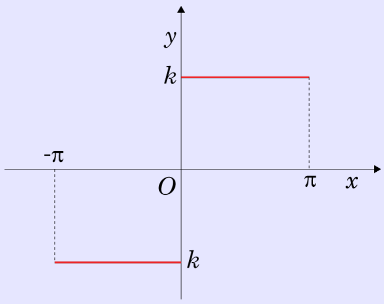

Figure 6.7: Figure with Example 217.

in the interval $(-\pi, \pi)$. Its analytic form is

$$f(x) = \begin{cases} -k, & \text{for} \quad -\pi < x < 0 \\ k & \text{for} \quad 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x). \tag{6.44}$$

Solution

From (6.11) it follows that $a_0 = 0$, and from (6.12) we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x =$$

$$\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\cos nx\,\mathrm{d}x\right] = \tag{6.45}$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0,$$

because $\sin nx = 0$ at $-\pi$, 0 and π, for each $n = 1, 2, \ldots,$. [7]

In the same way, from (6.12) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} k\sin nx\,\mathrm{d}x\right] =$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]. \tag{6.46}$$

Given that $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, we finally obtain

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos n\pi - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

$$\cos n\pi = \begin{cases} -1 & \text{for odd values of} \quad n, \\ 1 & \text{for even values of} \quad n. \end{cases}$$

Thus the Fourier coefficients for the given function have the following values

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \ldots,$$

that is

$$b_n = \begin{cases} \dfrac{4k}{n\pi}, & \text{for odd values of } n, \\ 0, & \text{for even values of } n. \end{cases}$$

As coefficients a_n are equal to zero, the corresponding Fourier series is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \ldots\right). \tag{6.47}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \quad S_2 = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x) \quad \text{etc.}$$

Graphs of partial sums are given in Figure 6.8 and they show how the series converges and how its sum is equal to the given function $f(x)$. At points $x = 0$ and $x = \pi$, which are the points of discontinuity of $f(x)$, all partial sums have a value of zero, which is the arithmetic mean of the values $-k$ and k.

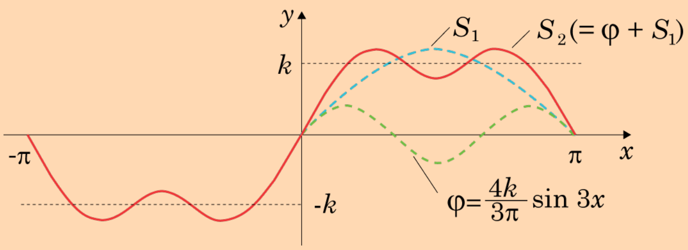

Figure 6.8:

Problem 218

Expand the function $f(x) = x$ into a Fourier series in the interval $(-\pi, \pi)$.

Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad x \in (-\pi, \pi).$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\frac{1}{2}x^2\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi^2 - \pi^2) = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx.$$

[7] Note that this result can be obtained directly, as the observed function is odd.

Bu introducing the substitution

$$nx = y \quad \Rightarrow \quad x = \frac{1}{n}y \quad dx = \frac{1}{n}dy$$

we obtain

$$a_n = \frac{1}{\pi n^2}\int_{-n\pi}^{n\pi} y\cos y dy.$$

Partial integration

$$y = u, du = dy, \cos y dy = dv, v = \sin y$$

yields

$$\begin{aligned} a_n &= \frac{1}{n^2\pi}\left(y\sin y\Big|_{-n\pi}^{n\pi} - \int_{-n\pi}^{n\pi}\sin y dy\right) = \\ &= \frac{1}{n^2\pi}\left(n\pi\sin n\pi - n\pi\sin(-n\pi) + \cos y\Big|_{-n\pi}^{n\pi}\right) = \\ &= \frac{1}{n^2\pi}(\cos n\pi - \cos(-n\pi)) = \frac{1}{n^2\pi}(\cos n\pi - \cos n\pi) = 0. \end{aligned}$$

For b_n, we obtain, in a similar way

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = \frac{1}{n^2\pi}\int_{-n\pi}^{n\pi} y\sin y dy = \\ &= \frac{1}{n^2\pi}\left(-y\cos y\Big|_{-n\pi}^{n\pi} + \int_{-n\pi}^{n\pi}\cos y dy\right) = \\ &= \frac{1}{n^2\pi}\left(-n\pi\cos n\pi - n\pi\cos(-n\pi) + \sin y\Big|_{-n\pi}^{n\pi}\right) = \\ &= \frac{1}{n^2\pi}(-2n\pi\cos n\pi + \sin n\pi + \sin n\pi) = \frac{1}{n^2\pi}(-2n\pi(-1)^n) = \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx. \tag{6.48}$$

R Note. The function $f(x) = x$ is odd and its expansion contains only coefficients next to the odd sine function.

Problem 219

Expand the function $f(x) = |x|$ into a Fourier series in the interval $(-\pi, \pi)$.

Solution

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\frac{1}{2}x^2\Big|_0^{\pi} = \pi,$$

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}|x|\cos nx\,dx = \frac{2}{\pi}\int\limits_{0}^{\pi}x\cos nx\,dx = \frac{2}{n^2\pi}\int_0^{n\pi}y\cos y\,dy =$$

$$= (y = u, du = dy, \cos y\,dy = dv, v = \sin y) =$$

$$= \frac{2}{n^2\pi}\left(y\sin y\Big|_0^{\pi} - \int\limits_0^{n\pi}\sin y\,dy\right) =$$

$$= \frac{2}{n^2\pi}\cos y\Big|_0^{n\pi} = \frac{2}{n^2\pi}((-1)^n - 1) =$$

$$= \begin{cases} 0 & \text{for even values of } n, \\ -\dfrac{4}{n^2\pi} & \text{for odd values of } n. \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nx\,dx = 0.$$

Thus, the Fourier series of the function $f(x) = |x|$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x.$$

Problem 220

Expand the following function into a Fourier series

$$f(x) = \begin{cases} ax, & -\pi < x \le 0, \\ bx, & 0 < x < \pi. \end{cases}$$

Solution

$$f(x) = \frac{\pi(b-a)}{4} + \frac{2(a-b)}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2} + (a+b)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}.$$

Problem 221

Expand the function $f(x) = \cos ax$ into a Fourier series in the interval $(-\pi, \pi)$, $(a \ne \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots)$.

Result

$$f(x) = \frac{2\sin\pi a}{\pi}\left[\frac{1}{2a} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{a\cos nx}{n^2 - a^2}\right].$$

Problem 222

Expand the function $f(x) = \sin ax$ into a Fourier series in the interval $(-\pi, \pi)$, $(a \neq 0, \pm 1, \pm 2, \dots)$

Result

$$f(x) = \frac{2\sin \pi a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2 - a^2}.$$

Problem 223

Expand the function $f(x) = \operatorname{sh} ax$ into a Fourier series in the interval $(-\pi, \pi)$.

Result

$$f(x) = \frac{2\operatorname{sh} \pi a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2 + a^2}.$$

Problem 224

Expand the function $f(x) = \operatorname{ch} ax$ into a Fourier series in the interval $(-\pi, \pi)$.

Result

$$f(x) = \frac{2\operatorname{sh} a\pi}{\pi} \left[\frac{1}{2a} + \sum_{n=1}^{\infty} (-1)^n \frac{a \cos nx}{a^2 + n^2} \right].$$

Problem 225

Expand the function $f(x) = e^{ax}$ into a Fourier series in the interval $(-l, l)$.

Result

$$f(x) = 2\operatorname{sh}(al) \cdot \left[\frac{1}{2al} + \sum_{n=1}^{\infty} (-1)^n \frac{al \cos nx - \pi n \sin nx}{(al)^2 + (\pi n)^2} \right].$$

Problem 226

Expand the function $f(x) = x\sin x$ into a Fourier series in the interval $(-\pi, \pi)$.

Result

$$f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2 - 1}\cos nx.$$

Problem 227

Expand the function $f(x) = x\cos x$ into a Fourier series in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Result

$$f(x) = \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(4n^2 - 1)^2}\sin 2nx.$$

Problem 228

Expand the function $f(x) = x(\pi - x)$ into a Fourier series in the interval $[0, \pi)$, $f(x) = f(x+\pi)$.

Result

$$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^2}.$$

Problem 229

Expand the function $f(x) = \dfrac{\pi - x}{2}$ into a Fourier series in the interval $(0, 2\pi)$.

Result

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Problem 230

Expand the following function into a Fourier series

$$f(x) = \begin{cases} A & 0 < x < l, \\ 0 & l < x < 2l, \end{cases}$$

in the interval $(0, 2l)$, where A is a constant.

Result

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin \frac{(2k+1)\pi x}{l}.$$

Problem 231

Expand the function $f(x) = \pi^2 - x^2$ into a Fourier series in the interval $(-\pi, \pi)$.

Result

$$f(x) = \frac{2}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$$

V

PDE

7. Partial differential equations

Introduction

Partial differential equations (PDE) occur in various problems of physics, geometry, and technology[1] in cases where the function describing a given process or phenomenon depends on two or more independent variables. Note that only simpler problems can be described by ordinary differential equations. Problems in the field of fluid mechanics, solid state mechanics, heat propagation, electromagnetism, etc., are described by partial differential equations.

The theory of partial differential equations is one of the best examples of the interconnectedness between mathematics and other scientific fields. Solving these equations is not only of formal mathematical interest but also a condition for understanding the process in the natural and other sciences. Thus, in the eyes of mathematicians, the solution of equations is a number or a function. Seen through the eyes of a physicist or biologist, a solution describes a process, and it thus has a physical meaning.

There are no general methods for solving second-order partial differential equations, unlike ordinary differential equations and first-order partial differential equations, with one unknown function.[2] The reason lies primarily in the fact that second-order partial differential equations appeared in mathematical physics and mechanics almost at the very beginning of the creation of mathematical analysis (more than two centuries ago). In addition, the solutions of these equations had to satisfy both the initial and boundary conditions, so they (solutions) were sought from problem to problem. The problems posed in this way attracted the attention of mathematicians, but they sought solutions for specific equations, as general methods were not sufficiently developed. However, it is interesting that even the development of science in recent times has not brought general methods for solving partial equations of the second order. It should

[1] Recently, they have also appeared in other fields of science, such as medicine, for mathematical modeling of the work of certain organs.

[2] Solving these equations comes down to integrating ordinary differential equations.

be emphasized that approximate (numerical) methods for solving PDE have advanced a lot in recent decades.

In this chapter, some of the more important partial differential equations that appear in technology will be discussed. We will develop some of these equations and show one of the methods for solving them (Fourier method of separation of variables). As solving a partial equation can become more complicated, when the initial or boundary conditions are changed, it is often resorted to solving these equations by numerical methods. Some of the numerical methods will be presented in the next chapter.

7.1 Definitions and notation

Definition

Let u be a function of variables $x_1, x_2, \ldots, x_n$, which has continuous partial derivatives up to order m, in the observed region $\Omega \subset \mathbb{R}^n$. Any relation between the variables x_i $(i = 1, \ldots, n)$, the function u and its partial derivatives

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^m u}{\partial x_1^m}, \ldots, \frac{\partial^m u}{\partial x_n^m}\right) = 0, \qquad (7.1)$$

is called a **partial differential equation**.

if the highest derivative in (7.1) is of order m ($m \leq n$), then the equation is called **partial differential equation of order** m.

Definition

Each function u, of variables $x_1, x_2, \ldots, x_k$, of which partial derivatives of the necessary order exist, and which identically satisfies, together with its partial derivatives, the equation (7.1), **is called a solution of the partial differential equation** (7.1).

Definition

The **general solution** of the equation (7.1) is the solution that contains the number of arbitrary independent functions equal to the order of the equation.

Definition

A **particular solution** is obtained from the general solution by assigning a specific form (expression) to the arbitrary functions.

Example 232

Observe the differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = 2x - y.$$

This is a partial differential equation of the second order, whose general solution is

$$u = x^2y - \frac{1}{2}xy^2 + F(x) + G(y).$$

In the special case, when $F(x) = 2\sin x$, $G(y) = 3y^4 - 5$, we obtain the particular solution

$$u = x^2y - \frac{1}{2}xy^2 + 2\sin x + 3y^4 - 5.$$

When solving partial differential equations, special conditions are often imposed, which the solution should satisfy. These conditions can be initial or boundary. They will be discussed in more detail later.

Monge notation

We will observe mainly second order partial differential equations, in which the unknown function u depend of two variables x and y. In that case, we will use the following, so called Monge[3] notation

$$\begin{aligned} p = u_x = \frac{\partial u}{\partial x}; \quad q = u_y = \frac{\partial u}{\partial y} \\ r = u_{xx} = \frac{\partial^2 u}{\partial x^2}; \quad s = u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \quad t = u_{yy} = \frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{7.2}$$

Using this notation, the general first order partial equation can be expressed in the form

$$F(x,y,u,p,q) = 0, \tag{7.3}$$

and the general second order partial equation as

$$F(x,y,u,p,q,r,s,t) = 0. \tag{7.4}$$

7.2 Formation of partial differential equations

Partial differential equations are formed in one of the following ways

- by elimination of variable functions,
- by elimination of constants, and
- by a mathematical description of a problem (in geometry, mechanics, physics, technology, etc.).

Let us demonstrate this on several examples.

[3]Gaspard Monge, comte de Péluse, 1746-1818, French mathematician, founder of the École polytechnique and one of the founders of the École normale. He offered a new approach to infinitesimal geometry, and was the author of a new method for geometric integration. He also obtained significant results in analytical geometry in the space.

Example 233

Let $f_1(\xi)$ and $f_2(\eta)$ be arbitrary differentiable functions, where $\xi = y + ax$, and $\eta = y - ax$. Determine the partial differential equation, whose solution satisfies the equation

$$z = f_1(\xi) + f_2(\eta).$$

Solution

Using the Monge notation, we obtain

$$p = \frac{\partial z}{\partial x} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f_2}{\partial \eta}\frac{\partial \eta}{\partial x} = a\frac{\mathrm{d}f_1}{\mathrm{d}\xi} - a\frac{\mathrm{d}f_2}{\mathrm{d}\eta} = a\left(f_1' - f_2'\right),$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f_2}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\mathrm{d}f_1}{\mathrm{d}\xi} + \frac{\mathrm{d}f_2}{\mathrm{d}\eta} = f_1' + f_2',$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(p) = a\left(\frac{\partial f_1'}{\partial x} - \frac{\partial f_2'}{\partial x}\right) = a\left(\frac{\mathrm{d}f_1'}{\mathrm{d}\xi}\frac{\partial \xi}{\partial x} - \frac{\mathrm{d}f_2'}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x}\right) =$$
$$= a^2\left(f_1'' + f_2''\right), \tag{7.5}$$
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(q) = \frac{\partial f_1'}{\partial y} + \frac{\partial f_2'}{\partial y} = \frac{\mathrm{d}f_1'}{\mathrm{d}\xi}\frac{\partial \xi}{\partial y} + \frac{\mathrm{d}f_2'}{\mathrm{d}\eta}\frac{\partial \eta}{\partial y} =$$
$$= f_1'' + f_2''. \tag{7.6}$$

From (7.5) and (7.6) we obtain the required partial differential equation

$$r - a^2 t = 0.$$

Example 234

Let a function be given in the form

$$z = a^2x^2 + 2abxy + b^2y^2 + cx + dy + e,$$

where a, b, c, d and e are arbitrary constants. Write the differential equation this function satisfies.

Solution

Starting from Monge notation, we obtain

$$p = \frac{\partial z}{\partial x} = 2a^2x + 2aby + c, \quad q = \frac{\partial z}{\partial y} = 2b^2y + 2abx + d,$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(p) = 2a^2, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(q) = \frac{\partial}{\partial y}(p) = 2ab,$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(q) = 2b^2.$$

From the last relations we obtain the required equation

$$rt - s^2 = 0.$$

Example 235

Determine the differential equation of the vibrating string (Fig. 5.1).

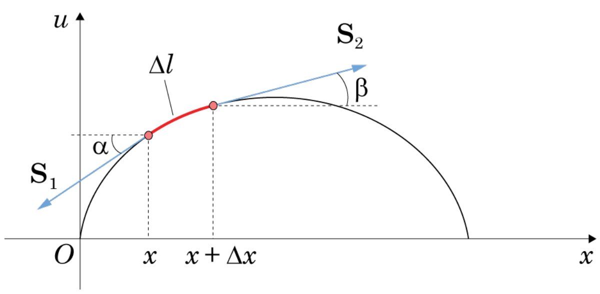

Figure 7.1: The vibrating string.

Solution

Assume the following

- the mass of the wire is constant per unit length (homogeneous wire),
- the weight of the wire is neglected (the force of gravity is negligible in relation to the forces that occur in the wire),
- the horizontal displacement of the points of the wire u is small in comparison with its length, so it can be considered that the points have only a vertical displacement. The deflections and inclinations at each point are also small.

We start from Newton's law

$$m\mathbf{a} = \sum_i \mathbf{S}_i,$$

where m is mass, $\mathbf{a}$ acceleration, and $\mathbf{S}_i$ forces acting on the observed point.

The projections of this vector equation (in this example there are two forces, thus

$i = 1, 2$) to the axes of the Cartesian coordinate system x and y are

$$ma_x = -S_1 \cos\alpha + S_2 \cos\beta = 0 \quad \Rightarrow \quad S_1 \cos\alpha = S_2 \cos\beta \equiv S, \quad \Rightarrow$$
$$\cos\alpha = \frac{S}{S_1}, \quad \cos\beta = \frac{S}{S_2}, \tag{7.7}$$

where we have used the assumption that there is no displacement in the direction of the x-axis ($a_x = 0$).

$$ma_u = -S_1 \sin\alpha + S_2 \sin\beta \quad \Rightarrow \quad S_2 \sin\beta - S_1 \sin\alpha = \rho V \frac{\partial^2 u}{\partial t^2},$$

$\frac{\partial^2 u}{\partial t^2}$ is the projection of acceleration a_u, and $u = u(x,t)$. Given that $m = \rho V$, and $V = \Delta\ell \cdot P$, where ρ is the density, $\Delta\ell$ the length, and P the cross-sectional area of the observed wire, and if we assume that P is the unit area, by dividing the second equation by S, we obtain

$$\frac{S_2}{S} \sin\beta - \frac{S_1}{S} \sin\alpha = \rho \frac{\Delta\ell}{S} \frac{\partial^2 u}{\partial t^2}.$$

Using (7.7), we further obtain

$$\text{tg}\beta - \text{tg}\alpha = \rho \frac{\Delta\ell}{S} \frac{\partial^2 u}{\partial t^2}. \tag{7.8}$$

Given that

$$\left.\frac{\partial u}{\partial x}\right|_x = \text{tg}\alpha \quad \text{i} \quad \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} = \text{tg}\beta,$$

and $\Delta\ell \approx \Delta x$, we obtain

$$\frac{\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x}{\Delta x} = \frac{\rho}{S} \frac{\partial^2 u}{\partial t^2}.$$

Observe now the limit value, when $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x \right) = \lim_{\Delta x \to 0} \frac{\rho}{S} \frac{\partial^2 u}{\partial t^2}.$$

As the left hand side represents the second partial derivative for x (by definition), and the function under the limit value on the right hand side does not depend on Δx, we finally obtain

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}, \tag{7.9}$$

where $c^2 = \rho / S$. The relation (7.9) represent the so-called one-dimensional **wave equation**.

Example 236

Find the differential equation of the family of spheres of radius r and center in the $x = y$ plane.

Solution

The equation of this family of spheres is

$$(x-a)^2+(y-a)^2+(z-b)^2=R^2. \tag{7.10}$$

Considering z to be a function of x and y, and differentiating the relation (7.10) with respect to x and y, respectively, yields

$$2(x-a)+2(z-b)\frac{\partial z}{\partial x}=0, \quad 2(y-a)+2(z-b)\frac{\partial z}{\partial y}=0,$$

or

$$(x-a)+(z-b)p=0, \quad (y-a)+(z-b)q=0.$$

Let us introduce the notation $z-b=-m$. It follows that $x-a=pm$, and $y-a=qm$. Substituting into the equation (7.10) yields

$$m^2(p^2+q^2+1)=R^2. \tag{7.11}$$

Given that $(x-y)=(p-q)m$, it follows that $m=\dfrac{x-y}{p-q}$. Further, substituting m obtained it this way into (7.11), we have

$$\left(\frac{x-y}{p-q}\right)^2(p^2+q^2+1)=R^2,$$

that is, the required partial differential equation

$$(x-y)^2(p^2+q^2+1)=R^2(p-q)^2.$$

Example 237

Determine the equation of the oscillating membrane.

R Before we start solving this task, let us explain the terms that appear in it:

- A membrane is a material surface, i.e. a geometric surface to which a continuously distributed mass is assigned.
- Density is the mass per unit area of the membrane. If the density is constant, then the membrane is called a homogeneous membrane.
- Stress is the force per unit length of the membrane.
- Slope is the angle that a tangent direction, at a point of the membrane, forms with its projection on the xy – plane.

Solution

Let us first state the physical conditions (constraints) under which the required equation is derived:

1° an elastic membrane is observed, whose density is constant.

2° The membrane is deformed so that its boundary remains fixed to the xy – plane. At such deformation, the stress in the membrane T is the same at all points and directions and it does not change over time.

3° The deflection of the membrane $u(x,y,t)$ (displacement of the points of the membrane in the z – axis direction) is small relative to membrane dimensions.

4° The slopes of the membrane at each point are small.

Imagine that we have cut a particle of the membrane ΔP (see Fig. 7.2). This part remains in the same state as before cutting, if the influence of the removed part is replaced by appropriate forces.

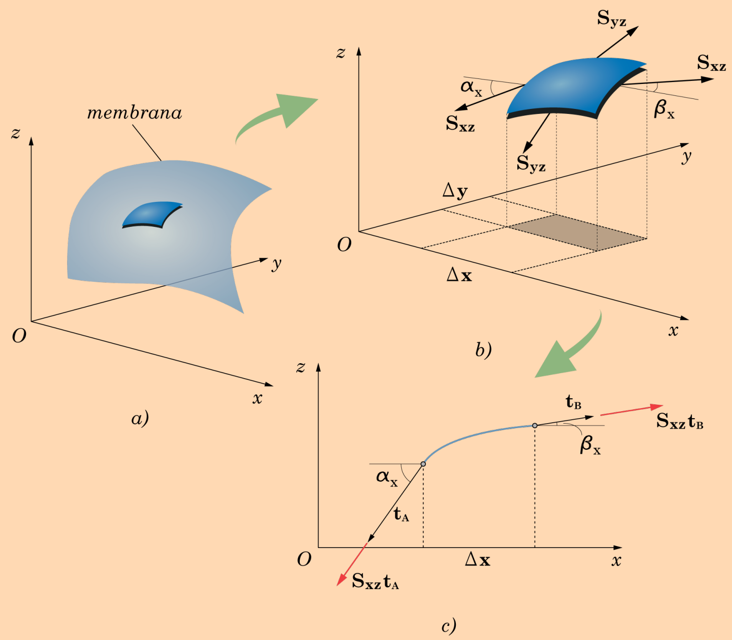

Figure 7.2: Membrane.

The projection of these forces on the xz and yz planes, respectively, expressed in terms of the stress T, can be written as

$$S_{xz} = T \cdot \Delta y, \quad \text{that is} \quad S_{yz} = T \cdot \Delta x.$$

Projection of the equation of motion ($\Delta m\mathbf{a} = \sum_i \mathbf{S}_i$) on the z direction is

$$\Delta m \frac{\partial^2 u}{\partial t^2} = S_{xz}(\sin\beta_x - \sin\alpha_x) + S_{yz}(\sin\beta_y - \sin\alpha_y) \tag{7.12}$$

where $\dfrac{\partial^2 u}{\partial t^2}$ is the projection of acceleration on the z – axis, and $\Delta m = \rho\Delta x\Delta y$.

As in this case, similarly to the case of the vibrating wire (according to assumption 4°), we have

$$\sin\beta_x \approx \mathrm{tg}\beta_x = \frac{\partial u(x+\Delta x, y)}{\partial x}, \quad \sin\alpha_x \approx \mathrm{tg}\alpha_x = \frac{\partial u(x, y)}{\partial x}$$

that is

$$\sin\beta_y \approx \mathrm{tg}\beta_y = \frac{\partial u(x, y+\Delta y)}{\partial y}, \quad \sin\alpha_y \approx \mathrm{tg}\alpha_y = \frac{\partial u(x, y)}{\partial y},$$

and thus the respective projections of these forces on the $z-$axis are

$$T\Delta y(\sin_x\beta - \sin_x\alpha) \approx T\Delta y(\mathrm{tg}_x\beta - \mathrm{tg}_x\alpha) = T\Delta y\left[\frac{\partial u(x+\Delta x, y)}{\partial x} - \frac{\partial u(x,y)}{\partial x}\right] \approx$$
$$\approx T\Delta x\Delta y\frac{\partial^2 u}{\partial x^2} + O_1(\Delta x, \Delta y),$$

where

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} O_1 = 0.$$

Similarly, for the other force we obtain

$$T\Delta x(\sin_y\beta - \sin_y\alpha) \approx T\Delta y(\mathrm{tg}_x\beta - \mathrm{tg}_y\alpha) = T\Delta x\left[\frac{\partial u(x, y+\Delta y)}{\partial y} - \frac{\partial u(x,y)}{\partial y}\right] \approx$$
$$\approx T\Delta x\Delta y\frac{\partial^2 u}{\partial y^2} + O_2(\Delta x, \Delta y),$$

where

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} O_2 = 0.$$

With these constraints, the equation (7.12) becomes

$$\rho\Delta x\Delta y\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\Delta u_x}{\Delta x} + \frac{\Delta u_y}{\Delta y}\right)\Delta x\Delta y + O_1 + O_2.$$

Observe now the limit process, when $\Delta x \to 0$ and $\Delta y \to 0$. As a result, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

We have thus obtained the partial differential equation for the oscillating membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\Delta u, \tag{7.13}$$

where $\frac{T}{\rho} = c^2$, and the operator Δ is the Delta operator.[4]

R Note that the constant c introduced in this way has the dimension of velocity, and the equation itself is also known as the **wave equation**. Unlike equation (7.9), which is called one-dimensional, this one is two-dimensional, because the function u depends on three variables, the first two being coordinates x and y, while the third variable is time t.

7.3 Linear and quasilinear first order PDE

A linear first order partial differential equation has the form

$$\sum_{i=1}^{n} a_i\frac{\partial u}{\partial x_i} + bu + c = 0, \tag{7.14}$$

where $u = u(x_1, x_2, \ldots, x_n)$ is the unknown function, and the coefficients a_i, b and c are, in the general case, functions of the form

$$\begin{aligned} a_i &= a_i(x_1, x_2, \ldots, x_n), \quad i = 1, \ldots, n, \\ b &= b(x_1, x_2, \ldots, x_n), \\ c &= c(x_1, x_2, \ldots, x_n). \end{aligned}$$

A quasilinear first order partial differential equation has the form

$$\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + b = 0. \tag{7.15}$$

In this case, coefficients next to respective derivatives are functions of the form

$$\begin{aligned} a_i &= a_i(x_1, x_2, \ldots, x_n, u), \quad i = 1, \ldots, n, \\ b &= b(x_1, x_2, \ldots, x_n, u). \end{aligned}$$

Thus, the equation is linear with respect to the first derivatives, but may be nonlinear with respect to the unknown function u.

7.3.1 On solutions for PDE

To integrate (solve) the equation (7.14), i.e. (7.15), means to find all functions $u(x_1, \ldots, x_n)$ that, together with their partial derivatives, identically satisfy the initial equation, for arbitrary values of the independent variables $x_1, \ldots, x_n$.

The solution of equations (7.14), i.e. (7.15), is also called the **integral surface** (in case of a function with two variables).

Complete, general, singular, mixed and particular solution

Integrals (solutions) of partial differential equations can depend on

- arbitrary constants (complete solution or complete integral), or
- arbitrary functions (general solution or general integral), or
- both arbitrary constants and arbitrary functions (mixed solution or mixed integral).

In addition, the following solutions also occur

- **particular solution**, which is obtained from the complete solution by substituting the initial conditions and
- **singular solution** or singular integral.[5]

Cauchy problem

The notion of the **Cauchy problem**, in the theory of first order partial equations, implies the determination of the general solution that passes through some predetermined curve. For example, find that solution of the partial equation

$$\frac{\partial f}{\partial x} = \psi(x, y, f),$$

which for $x = x_0$ becomes $f = \varphi(y)$.

Thus, the task is to determine the surface ($f(x,y)$) that passes through the given curve ($\varphi(y)$), which lies in a plane parallel to the yOz plane ($x = x_0$).

[5] This and other solutions will be discussed later in more detail.

7.3.2 A general method for integrating linear first order PDE. First integral

In this section we will show the relation between linear first order PDE and a system of ordinary differential equations of the form

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = f_i(x, y_1, \ldots, y_n), \quad i = 1, 2, \ldots, n, \tag{7.16}$$

where x is the independent variable, and $y_i = y_i(x)$ are unknown functions of the variable x.

Assume that the functions f_i $(i = 1, \ldots, n)$, which depend on $n+1$ independent variables $x, y_1, \ldots, y_n$, are differentiable in some closed region D, of the space $\mathbb{R}^{n+1}$. In the theory of ordinary differential equations, it has been proved (see, for example, [42]) that under the given assumptions, only one integral curve of the system (7.16) passes through a given point $M_0(x_0, y_1^0, \ldots, y_n^0)$, of the region D, i.e. that there exists an unambiguously defined set of functions $y_i(x)$ that represents the solution of the initial system (7.16), and satisfies the conditions $y_i(x_0) = y_i^0$, $i = 1, \ldots, n$.

The **first integral** of the system (7.16) is every function

$$\psi(x, y_1, \ldots, y_n)$$

that is not identically reduced to a constant when y_i is replaced by a set of solutions of system (7.16). It is common to call the function of the form

$$\psi(x, y_1, \ldots, y_n) = c \tag{7.17}$$

the first integral, where c is an arbitrary constant.

Theorem 23

The necessary and sufficient condition for the function $\psi(x, y_1, \ldots, y_n) = c$ to be the first integral of the system (7.16) is that it satisfies the linear first order partial differential equation

$$\frac{\partial \psi}{\partial x} + \sum_{i=1}^{n} f_i(x, y_1, \ldots, y_n) \frac{\partial \psi}{\partial y_i} = 0. \tag{7.18}$$

Proof

The condition is necessary. Assume that the function $\psi(x, y_1, \ldots, y_n)$ is the first integral of the system (7.16). Then this function, for the set y_i that represents the solutions of the initial system (7.16), is reduced to a constant, namely

$$\psi(x, y_1, \ldots, y_n) = c. \tag{7.19}$$

As c is a constant, it follows that the total differential of this function is

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x} + \sum_{i=1}^{n} \frac{\partial \psi}{\partial y_i} \frac{\mathrm{d}y_i}{\mathrm{d}x} = 0, \tag{7.20}$$

that is, using the relations (7.16),

$$\frac{\partial \psi}{\partial x} + \sum_{i=1}^{n} \frac{\partial \psi}{\partial y_i} f_i(x, y_1, \ldots, y_n) = 0. \tag{7.21}$$

Thus, the condition is necessary.
The condition is sufficient. If the function ψ satisfies the equation (7.21), then it also satisfies the equation (7.20), which means that it represents a first integral of the initial system (7.16), under the condition that the functions y_i, $i = 1, \ldots, n$, represent a set of solutions of the given system.

R Note that if the functions ψ_i $(i = 1, \ldots, m)$ are m first integrals of the initial system, then any differentiable function of the form

$$F(\psi_1, \ldots, \psi_m).$$

is also a first integral of this system.

7.3.3 Symmetrical form of a system of ordinary differential equations

Observe a system of ordinary differential equations (ODE) (7.16)

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = f_i(x, y_1, \ldots, y_n), \quad i = 1, 2, \ldots, n. \tag{7.22}$$

Solving this system for dx, we can write in the form

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y_1}{f_1} = \cdots = \frac{\mathrm{d}y_n}{f_n}. \tag{7.23}$$

If we now introduce the following substitutions

$$x = x_1,\, y_1 = x_2, \ldots,\, y_n = x_{n+1},$$

and then divide the relations (7.23) by some function X_1, which is not identical to zero in the observed region D, we obtain

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = \cdots = \frac{\mathrm{d}x_{n+1}}{X_{n+1}} \tag{7.24}$$

the so called **symmetrical form** of the system of ordinary differential equations (7.22). Here we have introduced the substitutions

$$X_{i+1} = f_i X_1, \quad i = 1, 2, \ldots, n.$$

The points $M_0(x_1^0, \ldots, x_{n+1}^0)$ at which $X_i(M_0) = 0$ are called **singular points** of the system (7.24).

R Note that the unambiguity of the solution is violated at these points, so they should not be taken as initial conditions.

7.3.4 General solution of the linear homogeneous first order PDE

Observe the linear homogeneous first order partial differential equation of the form

$$L[u] \equiv \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = 0. \tag{7.25}$$

Coefficients $a_i = a_i(x_1,\dots,x_n)$ are assumed to be differentiable functions in a region D of the n–dimensional space, and not annulled simultaneously at point $M_0(x_1^0,\dots,x_n^0)$.

As we have previously shown, we can assign to this partial equation a system of ordinary differential equations, in symmetric form

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{7.26}$$

According to the previous theorem, the first integrals $\psi_i = c_i$ of the system (7.26) are solutions of the partial equation (7.25). Thus, solving the initial partial equation (7.25) comes down to solving the system (7.26), i.e. finding its first integrals.

From the system (7.26) we can determine $n-1$ mutually independent first integrals

$$\psi_i(x_1,\dots,x_n) = c_i, \qquad i = 1,2,\dots,n-1, \tag{7.27}$$

where c_i are arbitrary constants.

We have already noted that if ψ_i $(i = 1,\dots,(n-1))$ are first integrals, and thus each differentiable function of the following form is also a first integral

$$F(\psi_1,\dots,\psi_{n-1}).$$

Thus, F represents the general solution of the partial differential equation (7.25). This can be easily proved.

Assume that the general solution of equation (7.25) has the form

$$u = F(\psi_1,\dots,\psi_{n-1}),$$

where F is an arbitrary function, and $\psi_i = c_i$ $(i = 1,\dots,n-1)$ are first integrals of the system (7.26). It follows that

$$\begin{aligned}
L[F] = {} & a_1\left(\frac{\partial F}{\partial \psi_1}\frac{\partial \psi_1}{\partial x_1} + \cdots + \frac{\partial F}{\partial \psi_{n-1}}\frac{\partial \psi_{n-1}}{\partial x_1}\right) + \\
& + a_2\left(\frac{\partial F}{\partial \psi_2}\frac{\partial \psi_2}{\partial x_2} + \cdots + \frac{\partial F}{\partial \psi_{n-1}}\frac{\partial \psi_{n-1}}{\partial x_2}\right) + \cdots + \\
& + a_n\left(\frac{\partial F}{\partial \psi_n}\frac{\partial \psi_n}{\partial x_n}\right) = \\
& = \frac{\partial F}{\partial \psi_1}\left(\underbrace{a_1\frac{\partial \psi_1}{\partial x_1} + \cdots + a_n\frac{\partial \psi_1}{\partial x_n}}_{L[\psi_1]}\right) + \cdots + \\
& + \frac{\partial F}{\partial \psi_{n-1}}\left(\underbrace{X_1\frac{\partial \psi_{n-1}}{\partial x_1} + \cdots + a_n\frac{\partial \psi_{n-1}}{\partial x_n}}_{L[\psi_{n-1}]}\right) = 0,
\end{aligned} \tag{7.28}$$

which was to be proved.

7.3.5 General solution of linear non-homogeneous first order PDE

The solution of the non-homogeneous first order partial differential equation can be obtained in a similar way as for the homogeneous case.

Observe the non-homogeneous first order partial differential equation

$$\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = b, \tag{7.29}$$

where u is an unknown continuous function of variables x_i, $i = 1,\dots,n$. The values a_i and b are assumed to be functions of the form $a_i = a_i(x_1,\dots,x_n)$ and $b = b(x_1,\dots,x_n,u)$.

Let us search for a general solution in the form

$$v(x_1,\dots,x_n,u) = 0. \tag{7.30}$$

By differentiating, we obtain

$$\frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial u}\frac{\partial u}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x_i} = -\frac{\dfrac{\partial v}{\partial x_i}}{\dfrac{\partial v}{\partial u}}. \tag{7.31}$$

Substituting $\dfrac{\partial u}{\partial x_i}$, from (7.31) into (7.29), we obtain

$$a_1\left(-\frac{\dfrac{\partial v}{\partial x_1}}{\dfrac{\partial v}{\partial u}}\right) + \cdots + a_n\left(-\frac{\dfrac{\partial v}{\partial x_n}}{\dfrac{\partial v}{\partial u}}\right) = b,$$

that is, multiplying by $-\dfrac{\partial v}{\partial u}$,

$$a_1\frac{\partial v}{\partial x_1} + \cdots + a_n\frac{\partial v}{\partial x_n} = -b\frac{\partial v}{\partial u},$$

or

$$a_1\frac{\partial v}{\partial x_1} + \cdots + a_n\frac{\partial v}{\partial x_n} + b\frac{\partial v}{\partial u} = 0. \tag{7.32}$$

This expression represents a homogeneous first order differential equation. Thus, any solution of equation (7.32), which contains the variable u, when equated to 0, gives the solution of equation (7.29) in the form (7.30).

To this equation corresponds the system of ordinary equations

$$\frac{\mathrm{d}x_1}{a_1} = \cdots = \frac{\mathrm{d}x_n}{a_n} = \frac{\mathrm{d}u}{b}. \tag{7.33}$$

First integrals are of the from

$$\begin{aligned} \psi_0(x_1,\dots,x_n,u) &= c_0, \\ \vdots &= \vdots \\ \psi_{n-1}(x_1,\dots,x_n,u) &= c_{n-1}. \end{aligned} \tag{7.34}$$

The general integral is of the form

$$F(\psi_0,\dots,\psi_{n-1}) = 0, \tag{7.35}$$

where F is a differentiable function of its arguments.

7.3.6 Pfaffian equation

Pfaffian equation is the equation of the form

$$P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0, \tag{7.36}$$

where $z = z(x,y)$ is the unknown function, and P, Q and R are given, continuously differentiable functions

$$P = P(x,y,z), \quad Q = Q(x,y,z), \quad R = R(x,y,z),$$

in 3–dimensional space $\mathbb{R}^3$. The equation (7.36) can be relatively easily integrated in two cases:

1° when the left hand side represents a total differential of some function, which we shall denote by u, and

2° when there is such a function (integration factor) by which the Pfaffian equation needs to be multiplied to obtain a total differential.

In the first case, observe some function $u(x,y,z)$, whose total differential is

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z.$$

By comparison with (7.36), in order to be a total differential, this expression must satisfy the following conditions

$$\frac{\partial u}{\partial x} = P, \qquad \frac{\partial u}{\partial y} = Q, \qquad \frac{\partial u}{\partial z} = R. \tag{7.37}$$

If u is a twice continuously differentiable function ($u \in C^2(D)$), then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}, \quad \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial^2 u}{\partial x \partial z}. \tag{7.38}$$

From (7.37) and (7.38) we obtain

$$\frac{\partial}{\partial x}(Q) = \frac{\partial}{\partial y}(P), \quad \frac{\partial}{\partial y}(R) = \frac{\partial}{\partial z}(Q), \quad \frac{\partial}{\partial z}(P) = \frac{\partial}{\partial x}(R), \tag{7.39}$$

the integrability conditions for the initial equation (7.36). Thus, in this case, the equation (7.36) can be written as

$$\mathrm{d}u = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0 \tag{7.40}$$

from where, by integrating, we obtain the implicit solution of the initial equation

$$u(x,y,z) = \int_{x_0}^{x} P\mathrm{d}x + \int_{y_0}^{y} Q\mathrm{d}y + \int_{z_0}^{z} R\mathrm{d}z = c, \tag{7.41}$$

where c is an arbitrary constant.

In the second case, assume that there exists a function $v(x,y,z)$, such that $\mathrm{d}u$, defined by the relation

$$\mathrm{d}u = vP\mathrm{d}x + vQ\mathrm{d}y + vR\mathrm{d}z = 0$$

is a total differential.

Let us now look for the conditions that the functions P, Q and R must satisfy in order for the function v to exist, as well as for the relation from which we could determine the function v.

Similarly to the previous case, we have the conditions

$$\frac{\partial u}{\partial x} = vP, \qquad \frac{\partial u}{\partial y} = vQ, \qquad \frac{\partial u}{\partial z} = vR, \tag{7.42}$$

that is
$$\frac{\partial}{\partial x}(vQ)=\frac{\partial}{\partial y}(vP),\quad \frac{\partial}{\partial y}(vR)=\frac{\partial}{\partial z}(vQ),\quad \frac{\partial}{\partial z}(vP)=\frac{\partial}{\partial x}(vR),$$
from where we obtain
$$\begin{aligned}
Q\frac{\partial v}{\partial x}+v\frac{\partial Q}{\partial x}=v\frac{\partial P}{\partial y}+P\frac{\partial v}{\partial y} \quad &\Rightarrow \quad v\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=P\frac{\partial v}{\partial y}-Q\frac{\partial v}{\partial x},\\
R\frac{\partial v}{\partial y}+v\frac{\partial R}{\partial y}=v\frac{\partial Q}{\partial z}+Q\frac{\partial v}{\partial z} \quad &\Rightarrow \quad v\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)=Q\frac{\partial v}{\partial z}-R\frac{\partial v}{\partial y},\\
P\frac{\partial v}{\partial z}+v\frac{\partial P}{\partial z}=v\frac{\partial R}{\partial x}+R\frac{\partial v}{\partial x} \quad &\Rightarrow \quad v\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)=R\frac{\partial v}{\partial x}-P\frac{\partial v}{\partial z}.
\end{aligned}$$
These equations are linear partial equations of the first order. As we have shown earlier, the following systems of ordinary differential equations can be unambiguously assigned to them, and thus, from the first equation we obtain
$$\frac{\mathrm{d}x}{-Q}=\frac{\mathrm{d}y}{P}=\frac{\mathrm{d}v}{v\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)} \quad \Rightarrow \quad \frac{\mathrm{d}v}{v}=\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{-Q}\mathrm{d}x=\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}\mathrm{d}y. \tag{7.43}$$
Similarly, from the second equation we obtain
$$\frac{\mathrm{d}v}{v}=\frac{\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}}{-R}\mathrm{d}y=\frac{\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}}{Q}\mathrm{d}z, \tag{7.44}$$
and from the third
$$\frac{\mathrm{d}v}{v}=\frac{\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}}{R}\mathrm{d}x=\frac{\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}}{-P}\mathrm{d}z. \tag{7.45}$$
Given that the function on the left hand side of these equations is the same, it follows that the coefficients next to the corresponding differentials are equal, i.e.
$$\begin{aligned}
\frac{-\frac{\partial Q}{\partial x}+\frac{\partial P}{\partial y}}{Q}&=\frac{\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}}{R},\\
\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}&=\frac{\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}}{-R},\\
\frac{\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}}{Q}&=\frac{\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}}{-P}.
\end{aligned}$$
From here we obtain
$$\begin{aligned}
Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)&=R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right),\\
R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)&=P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right),\\
P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)&=Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right).
\end{aligned}$$

Adding the left hand and right hand sides of these relations, respectively, we further obtain

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=$$
$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right).$$

From the last equation we obtain

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0, \tag{7.46}$$

namely, the **condition of integrability**. Thus, there exists a function v, by which (7.36) needs to be multypelied in order to be represented as a total differential

$$\mathrm{d}u = v\,(P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z)=0.$$

The condition of integrability of the Pfaffian equation (7.46) can be represented in a more suitable form. Namely, the expressions next to P, Q and R, in equation (7.46), represent components of a rotor of the vector $\mathbf{v}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ (see definition (4.46), on p. 92). The expression (7.46) itself, represents now a scalar product (see (1.44), p. 31)

$$\mathbf{v}\cdot \mathrm{rot}\mathbf{v}=0. \tag{7.47}$$

R Note. Given that P, Q and R are not identically equal to zero (otherwise we would have the identity 0=0), it follows that this condition (7.47) is satisfied either when $\mathrm{rot}\mathbf{v}=0$, or when vectors $\mathbf{v}$ and $\mathrm{rot}\mathbf{v}$ are orthogonal.

R Note. If the conditions for integrability (7.47) are satisfied, then the function v can be determined form one of the three equations (7.43) – (7.45).

Let us now outline a procedure for determining the solution of the equation (7.36), when the condition (7.47) is satisfied, and $\mathrm{rot}\mathbf{v}\neq 0$.

Assume that one variable, say z, is constant. The the observed equation becomes

$$P(x,y,z)\mathrm{d}x+Q(x,y,z)\mathrm{d}y=0, \tag{7.48}$$

namely, an ordinary differential equation, in which z has the role of a parameter. The solution of this equation is a function of the form

$$u(x,y,z)=c(z). \tag{7.49}$$

The arbitrary "constant" is a function of parameter z. The function $c(z)$ is selected so that the equation (7.36) is satisfied.

By differentiating the equation (7.49) we obtain

$$\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z-\frac{\mathrm{d}c}{\mathrm{d}z}\mathrm{d}z=0,$$

that is

$$\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\left(\frac{\partial u}{\partial z}-\frac{\mathrm{d}c}{\mathrm{d}z}\right)\mathrm{d}z=0. \tag{7.50}$$

Coefficients next to (7.50) and (7.36) must be proportional, so we obtain

$$\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z} - c'}{R}.$$

We can determine c' from this system of equations, for example from

$$\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial z} - c'}{R}. \tag{7.51}$$

Namely, it can be shown that if conditions $\mathbf{v} \cdot \text{rot}\mathbf{v} = 0$ and $\text{rot}\mathbf{v} \neq 0$ ($\mathbf{v} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$) are satisfied, the equation (7.51) depends only on z, $c'(z)$ and $u(x,y,z) = c(z)$.

7.3.7 Nonlinear first order PDE. Lagrange-Charpit method

A nonlinear first order differential equation (of two variables) is an equation of the form

$$f(x,y,z,p,q) = 0, \tag{7.52}$$

where $z = z(x,y)$ is the unknown function (of two independent variables x,y), $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. It is assumed that the function f, and all its partial derivatives, are continuous functions for all arguments, up to the required order.

As we have already mentioned, there are three types of solutions: complete, singular and general.

Complete and singular solutions

Assume that the equation (7.52) can be obtained by elimination of two arbitrary constants (parameters), say a and b, from the function

$$g(x,y,z,a,b) = 0. \tag{7.53}$$

Then the function g is called the **complete solution** (integral) of the partial equation (7.52).

R Note. Geometrically, the complete solution defined in this way represents a two-parameter family of surfaces, which may or may not have an envelope. The envelope of these surfaces is also a solution of equation (7.52) and it is called the **singular solution**.

The envelope (if it exists!) can be determined by eliminating constants a and b from the system of equations

$$g = 0, \quad \frac{\partial g}{\partial a} = 0, \quad \frac{\partial g}{\partial b} = 0. \tag{7.54}$$

If by eliminating constants a and b from this system we obtain a function

$$h(x,y,z) = 0, \tag{7.55}$$

which satisfies the initial equation (7.52), then $h(x,y,z)$ is called the **singular solution** (integral).

If the function $h(x,y,z)$ can be represented in the form of a product

$$h(x,y,z) = \xi(x,y,z) \cdot \eta(x,y,z), \tag{7.56}$$

where $\xi = 0$ satisfies the equation (7.52), while $\eta = 0$ does not, then $\xi = 0$ is a singular solution.

A singular solution can be obtained from a partial equation by eliminating p and q from the system

$$f = 0, \quad \frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0.$$

General solution

If the parameters a and b are not independent, that is, if, for example, $b = b(a)$, then the surface envelope is defined by the relation

$$g(x,y,z,a,b(a)) = 0, \quad \frac{\partial g}{\partial a} + \frac{\partial g}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}a} = 0. \tag{7.57}$$

This solution is called the **general solution** (general integral).

Note. If one complete solution of equation (7.52) is known, a general and singular solution (if it exists) can be obtained from it, by simply differentiating and eliminating the appropriate parameters. Let us demonstrate this claim.

Assume that the complete integral of the equation (7.52) is the function (7.53)

$$g(x,y,z,a,b) = 0. \tag{7.58}$$

Let parameters a and b be functions of x and y. Then, differentiating (7.58) first by x, and then by y, we obtain

$$\begin{aligned} \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} &= 0, \\ \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}q + \frac{\partial g}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial y} &= 0. \end{aligned} \tag{7.59}$$

Parameters a and b are determined from the condition

$$\begin{aligned} \frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} &= 0, \\ \frac{\partial g}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial y} &= 0. \end{aligned} \tag{7.60}$$

From this homogeneous system of equations we can determine $\frac{\partial g}{\partial a}$ and $\frac{\partial g}{\partial b}$. Then we have two different cases:

- the system determinant is not equal to zero, i.e.

$$\begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} \\ \frac{\partial a}{\partial y} & \frac{\partial b}{\partial y} \end{vmatrix} \neq 0,$$

  and the system has only trivial solutions

$$\frac{\partial g}{\partial a} = \frac{\partial g}{\partial b} = 0. \tag{7.61}$$

  From these equations we determine a and b as functions of x and y and obtain

$$g = g(x,y,z,a(x,y),b(x,y)),$$

  which represents the singular solution.
- The system determinant is equal to zero

$$\begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} \\ \frac{\partial a}{\partial y} & \frac{\partial b}{\partial y} \end{vmatrix} = 0,$$

  from where two possibilities follow

1° either

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial x} = \frac{\partial a}{\partial y} = \frac{\partial b}{\partial y} = 0,$$

and we obtain that a and b are constant, that is, that the function $g(x,y,z,a,b)$ is the **complete solution**,

2° or $b = \varphi(a)$, where φ is an arbitrary function. In that case

$$\frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial a}\frac{\partial a}{\partial x} = \left(\frac{\partial g}{\partial a} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial a}\right)\frac{\partial a}{\partial x} = 0,$$

that is, provided that $\dfrac{\partial a}{\partial x} \neq 0$,

$$\frac{\partial g}{\partial a} + \frac{\partial g}{\partial b}\varphi'(a) = 0, \tag{7.62}$$

to which both equations (7.37) are reduced. From (7.62) we can express $a = \psi(x,y)$, so we obtain

$$b = \varphi(\psi(x,y)) = \mu(x,y).$$

which is the **general solution**.

As can be concluded from the previous presentation, both singular and general solutions (if they exist) can be obtained from the complete solution of the first order partial differential equation. Therefore, the basic task is to find a complete solution. It can be determined by applying the Lagrange-Charpit[6] method.

Lagrange-Charpit method

Observe the general first order partial differential equation (of two independent variables x and y)

$$f(x,y,z,p,q) = 0. \tag{7.63}$$

The main idea of the Lagrange-Charpit method in finding the complete solution is to determine a function

$$g(x,y,z,p,q) = c_1 \tag{7.64}$$

that is functionally independent of f, where c_1 is an arbitrary constant. The function g should be such that from the system of partial equations

$$f = 0, \qquad g = c_1 \tag{7.65}$$

p and q can be calculated, i.e.

$$p = \varphi(x,y,z,c_1), \qquad q = \psi(x,y,z,c_1). \tag{7.66}$$

From the assumption that p and q can be determined from (7.65), it follows that

$$D_{pq} = \begin{vmatrix} \dfrac{\partial f}{\partial p} & \dfrac{\partial f}{\partial q} \\[2ex] \dfrac{\partial g}{\partial p} & \dfrac{\partial g}{\partial q} \end{vmatrix} \neq 0. \tag{7.67}$$

[6] Charpit

Further, given that $z = z(x,y)$, its differential

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y = p\mathrm{d}x + q\mathrm{d}y \tag{7.68}$$

represents the Pfaffian equation

$$\varphi(x,y,z,c_1)\mathrm{d}x + \psi(x,y,z,c_1)\mathrm{d}y - \mathrm{d}z = 0. \tag{7.69}$$

In order for this equation to be integrable, i.e. reduced to a complete differential, it is necessary to satisfy the condition of integrability (7.46), which in this case is reduced to ($P = p$, $Q = q$, $R = -1$)

$$\left(\frac{\partial p}{\partial y} + q\frac{\partial p}{\partial z}\right) - \left(\frac{\partial q}{\partial x} + p\frac{\partial q}{\partial z}\right) = 0. \tag{7.70}$$

By integrating the equation (7.69), another arbitrary constant is obtained, which finally yields a complete solution in the form

$$v = v(x,y,z,c_1,c_2).$$

Let us now find the relations and constraints from which we could determine the function g. To that end, we shall first define some concepts.

Definition

Two functions f and g are said to be in **involution** if $[f,g] = 0$.

The notation $[f,g]$ introduced here is determined by the expression

$$[f,g] = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial x} + p\frac{\partial g}{\partial z} \end{vmatrix} + \begin{vmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial y} + q\frac{\partial g}{\partial z} \end{vmatrix}. \tag{7.71}$$

The expression (7.71) is known in literature as **Mayer**[7] **bracket** .

In particular, if functions f and g do not depend explicitly of z, i.e. $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = 0$, the Mayer bracket comes down to

$$(f,g) = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial x} \end{vmatrix} + \begin{vmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial y} \end{vmatrix}. \tag{7.72}$$

The expression (f,g), defined by the relation (7.72) is known in literature as **Poisson bracket**.[8]

Theorem 24

Functions p and q, determined by equations (7.66) form a total differential (7.69) iff the functions f and g are in involution.

[7] Mayer

[8] Denis Poisson (1781-1840), French mathematician. He dealt with rational mechanics, probability calculus and mathematical physics. He laid the foundations of magnetism.

Proof

The condition is necessary. For the relation (7.69) to be a total differential, the conditions (7.70) must be satisfied

$$\left(\frac{\partial p}{\partial y}+q\frac{\partial p}{\partial z}\right)-\left(\frac{\partial q}{\partial x}+p\frac{\partial q}{\partial z}\right)=0. \tag{7.73}$$

Further, by differentiating the functions f and g by x and y, respectively, we obtain

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0, \tag{7.74}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0, \tag{7.75}$$

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial g}{\partial q}\frac{\partial q}{\partial x}=0, \tag{7.76}$$

$$\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial g}{\partial q}\frac{\partial q}{\partial y}=0. \tag{7.77}$$

Multiplying (7.74) by $\frac{\partial g}{\partial p}$, (7.75) by $\frac{\partial g}{\partial q}$, (7.76) by $-\frac{\partial f}{\partial p}$, (7.77) by $-\frac{\partial f}{\partial q}$, and then adding, yields

$$[f,g]+D_{pq}\left[\left(\frac{\partial p}{\partial y}+q\frac{\partial p}{\partial z}\right)-\left(\frac{\partial q}{\partial x}+p\frac{\partial q}{\partial z}\right)\right]=0. \tag{7.78}$$

Given that, according to the condition (7.73), the second member of the sum is equal to zero, it follows that

$$[f,g]=0,$$

that is, the functions f and g are in involution.
The condition is sufficient. If the functions f and g are in involution, then $[f,g]=0$, and thus from (7.78) it follows that the condition (7.73) is satisfied.

The task was to find the conditions that should be satisfied by the function $g=c_1$, so that we can determine from the system

$$f=0, \qquad g=c_1$$

the values of p and q, but so that the expression

$$p\mathrm{d}x+q\mathrm{d}y-\mathrm{d}z=0$$

is the complete differential of some function v ($\mathrm{d}v=p\mathrm{d}x+q\mathrm{d}y-\mathrm{d}z$). Based on the previous Theorem, we can see that the functions f and g must be in involution, i.e.

$$[f,g]=0.$$

Using the definition (7.71), this condition, in expanded form, comes down to

$$\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial g}{\partial y}+\left(\frac{\partial f}{\partial p}p+\frac{\partial f}{\partial q}q\right)\frac{\partial g}{\partial z}-$$
$$-\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p\right)\frac{\partial g}{\partial p}-\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q\right)\frac{\partial g}{\partial q}=0. \tag{7.79}$$

The last expression represents a linear first order partial equation of an unknown function g. As we have shown earlier, solving this equation comes down to solving a system of ordinary equations:

$$\frac{dx}{\dfrac{\partial f}{\partial p}} = \frac{dy}{\dfrac{\partial f}{\partial q}} = \frac{dz}{\dfrac{\partial f}{\partial p}p + \dfrac{\partial f}{\partial q}q} = \frac{dp}{-\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial z}p} = \frac{dq}{-\dfrac{\partial f}{\partial y} - \dfrac{\partial f}{\partial z}q},$$

or, if we introduce the notation

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_z = \frac{\partial f}{\partial z}, \quad f_p = \frac{\partial f}{\partial p}, \quad f_q = \frac{\partial f}{\partial q},$$

the previous expression can be written in the form

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}. \tag{7.80}$$

This system yields $f(x,y,z,p,q) = 0$ as one of the first integrals. Thus, at least another first integral of the same system must be found, in order to determine the values of p and q and obtain the total differential (7.69).

7.4 Linear second order PDE

A linear non-homogeneous second order partial differential equation is an equation of the form

$$L(u) \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + bu = c \tag{7.81}$$

where L denotes the linear operator, $u = u(x_1, \ldots, x_n)$ is the unknown function, and the coefficients are functions of the form

$$\begin{aligned} a_{ij} = a_{ij}(x_1, \ldots, x_n); \quad a_i = a_i(x_1, \ldots, x_n); \\ b = b(x_1, \ldots, x_n); \quad c = c(x_1, \ldots, x_n), \quad \text{where} \quad a_{ij} = a_{ji}. \end{aligned} \tag{7.82}$$

R Note that the assumption on the symmetry of coefficients a_{ij} is not a constraint, because for continuous functions $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \dfrac{\partial^2 u}{\partial x_j \partial x_i}$.

A linear homogeneous second order partial differential equation is an equation of the form

$$L(u) \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + bu = 0. \tag{7.83}$$

We can write these equations in a shorter form

$$L(u) = c, \quad \text{that is,} \quad L(u) = 0. \tag{7.84}$$

7.4.1 Some properties of homogeneous second order partial LDE

Property 1.

If $u_i(x_1,x_2)$ $(i=1,2,\dots,n)$ are solutions of homogeneous equation (7.83), then their arbitrary linear combination

$$u=\sum_{i=1}^{m} C_i u_i(x_1,x_2), \tag{7.85}$$

where C_i are arbitrary constants, is also a solution of equation (7.83).

Proof:

$$L(u)=L\left(\sum_{i=1}^{n} C_i u_i\right)=\sum_{i=1}^{n} L(C_i u_i)=\sum_{i=1}^{n} C_i L(u_i)=0. \tag{7.86}$$

If $u_1,u_2,\dots,u_n,\dots$ are solutions of the initial equation (7.83), and $C_1,C_2,\dots,C_n,\cdots$ are constants, then the function u given by the infinite series

$$u=\sum_{i=1}^{\infty} C_i u_i, \quad \text{in } \Omega \tag{7.87}$$

is also a solution of equation (7.83). Understandably, under the condition that the series on the right had side of the equation (7.87), as well as the series obtained by differentiating it formally member by member, including all possible second order derivatives, are convergent in Ω.

Such a solution u is said to be obtained by **superposition from** solutions u_i.

Property 2.

If $u_o(x_1,x_2,\alpha_1,\alpha_2)$ is a solution of equation (7.83), where α_i are parameters independent of x_i, then the following functions are also solutions

$$u=\int C(\alpha_1)\cdot u_o(x_1,x_2,\alpha_1)\,\mathrm{d}\alpha_1, \tag{7.88}$$

$$u=\iint C(\alpha_1,\alpha_2)\cdot u_o(x_1,x_2,\alpha_1,\alpha_2)\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2, \tag{7.89}$$

where $C(\alpha_1)$ and $C(\alpha_1,\alpha_2)$ are arbitrary functions, and is is assumed that the integrals above can be differentiated.

This solution, obtained from solution u_o, is said to be obtained **by integration**, by parameter α_1, that is, by parameters α_1,α_2.

If we assume that a_{ij}, a_i and b are constants, then the following properties are obtained

Property 3.

If $u_1(x_1,x_2)$ is a solution of the equation (7.83), then the function

$$u=u_1(x_1-\alpha_1,x_2-\alpha_2) \tag{7.90}$$

is also a solution of the equation.

This property can be proved by introducing the substitution

$$x_i=\xi_i+\alpha_i,$$

in the differential equation (7.83).

In this case it is said that the solution us obtained from the solution u_1 – **by shifting the argument**.

Property 4.

Combining the last two properties, we obtain that the solutions are also

$$u = \int C(\alpha_1)\, u_1\,(x_1 - \alpha_1, x_2)\, \mathrm{d}\alpha_1, \tag{7.91}$$

$$u = \iint C(\alpha_1, \alpha_2)\, u_1\,(x_1 - \alpha_1, x_2 - \alpha_2)\, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2. \tag{7.92}$$

For such solutions, it is said that they are obtained by **convolution** or as a resultant of functions C and u_1.

Property 5.

If the equation (7.83), with real constant coefficients, has a complex solution of the form

$$u = P(x_1, x_2) + i \cdot Q(x_1, x_2), \tag{7.93}$$

where P and Q are real functions, and $i = \sqrt{-1}$ is the imaginary unit, then the functions P and Q are also solutions of this equation.

Proof

$$L(u) = L(P + iQ) = L(P) + iL(Q) = 0 \quad \Rightarrow \quad L(P) = 0 \wedge L(Q) = 0. \tag{7.94}$$

7.4.2 Classification of second order LDE with two variables

When studying partial equations, the question arises as to whether it is possible to simplify the initial equation by introducing appropriate transformations.

In this chapter, we take, for simplicity, that $n = 2$, i.e. that the unknown function is of the form $u = u(x_1, x_2)$.

Observe now the homogeneous partial differential equation

$$L(u) = 0, \tag{7.95}$$

which will be transformed by introducing new variables ξ_1 and ξ_2

$$\xi_i = \xi_i(x_1, x_2), \quad J\left(\frac{\xi_1, \xi_2}{x_1, x_2}\right) \equiv \begin{vmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_1}{\partial x_2} \\ \dfrac{\partial \xi_2}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_2} \end{vmatrix} \neq 0, \quad i = 1, 2. \tag{7.96}$$

where J is the Jacobi[9] functional determinant or Jacobian.

The relation between new and old derivatives is given by the following equations

$$\frac{\partial u}{\partial x_j} = \sum_{i=1}^{n(=2)} \frac{\partial u}{\partial \xi_i} \frac{\partial \xi_i}{\partial x_j}, \quad j = 1, 2 \tag{7.97}$$

[9]Carl Gustav Jacobi 1804-1851, German mathematician. Author of important works in analysis, especially theory of elliptic functions.

$$\frac{\partial^2 u}{\partial x_k \partial x_l} = \sum_{i,j=1}^{n(=2)} \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} + \frac{\partial u}{\partial \xi_i} \frac{\partial^2 \xi_i}{\partial x_k \partial x_l} \right), \quad \begin{array}{l} k=l=1, \\ k=1, l=2 \\ k=l=2 \end{array} \tag{7.98}$$

Now the transformed equation (7.83) becomes

$$\bar{L} \equiv \sum_{i,j=1}^{n(=2)} \bar{a}_{ij} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{n(=2)} \bar{a}_i \frac{\partial u}{\partial \xi_i} + \bar{b}u = 0. \tag{7.99}$$

The relation between new and old coefficients is given by the following equations

$$\bar{a}_{kl} = \sum_{i,j=1}^{n(=2)} a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}, \quad \bar{a}_k = \sum_{i=1}^{n(=2)} a_i \frac{\partial \xi_k}{\partial x_i} + \sum_{i,j=1}^{n(=2)} a_{ij} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j}. \tag{7.100}$$

The general form of equation (7.83) is an equation that is linear only with respect to second derivatives

$$\sum_{i,j=1}^{n(=2)} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b = 0, \tag{7.101}$$

where coefficients b and a_{ij} are function of the form

$$b = b(x_1, x_2, u, \partial u/\partial x_1, \partial u/\partial x_2); \quad a_{ij} = a_{ij}(x_1, x_2). \tag{7.102}$$

The transformed form is

$$\sum_{i,j=1}^{n(=2)} \bar{a}_{ij} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \bar{b} = 0. \tag{7.103}$$

We choose new variables so that the transformed equation is as simple as possible, for example, that some of the coefficients $\bar{a}_{ij}$ are equal to zero. To this end, and considering the relations between the new and old coefficients (7.100), we observe the following first order partial equation

$$\sum_{i,j=1}^{n(=2)} a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} = 0, \tag{7.104}$$

which, given that $a_{12} = a_{21}$, can be represented also as the following set of two linear partial equations

$$\frac{\partial z}{\partial x_1} = \frac{-a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \frac{\partial z}{\partial x_2}. \tag{7.105}$$

To these equations corresponds the system of ordinary differential equations

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{-a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}, \tag{7.106}$$

or

$$a_{22} \cdot \mathrm{d}x_1^2 - 2a_{12} \cdot \mathrm{d}x_1 \mathrm{d}x_2 + a_{11} \cdot \mathrm{d}x_2^2 = 0. \tag{7.107}$$

Equations (7.106), that is, (7.107), are called **characteristic equations** of the partial differential equation (7.101).

Classification

In region S, in which coefficients a_{ij} and b are defined, observe a point $M(x_1,x_2)$ for which the following conditions are fulfilled:

1° discriminant $D \equiv a_{12}^2 - a_{11} \cdot a_{22} > 0$, i.e. the equation (7.107) has two different real solutions. In that case the equation (7.101) is said to be of **hyperbolic type** at point M;

2° if the $D < 0$, then the respective equation has conjugate complex solutions, and it is said that the equation is of **elliptic type** at point M;

3° if discriminant $D = 0$, then the respective equation has a double real solution, and it is said that the equation is of **parabolic type** at point M.

R Note that the type of the equation (7.101), in a region S or at a point M in that region, is invariant with respect to the transformation

$$\xi_i = \xi_i(x_1,x_2), \quad J \neq 0. \tag{7.108}$$

Namely, starting from relations (7.100), we can obtain

$$\bar{D} = D \cdot J^2. \tag{7.109}$$

Here, $\bar{D} \equiv \bar{a}_{12}^2 - \bar{a}_{11} \cdot \bar{a}_{22}$, and the discriminants (new-old) are thus of the same sign.

7.4.3 Reduction to canonical form

Reduction of hyperbolic equation to canonical form

In this case $D > 0$, which yields two solutions of the characteristic equation

$$\xi_i(x_1,x_2) = C_i, \quad (i = 1,2), \tag{7.110}$$

where C_i are arbitrary constants. We chose transformations in the form

$$\xi_i = \xi_i(x_1,x_2), \quad (i = 1,2), \tag{7.111}$$

and thus, in this case

$$\bar{a}_{11} = \bar{a}_{22} = 0. \tag{7.112}$$

The transformed equation (7.103) becomes

$$\frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} = F(\xi_1, \xi_2, u, \partial u/\partial \xi_1, \partial u/\partial \xi_2), \tag{7.113}$$

where $F = -\dfrac{\bar{b}}{2\bar{a}_{12}}$. This is the **canonical form** of the equation of hyperbolic type.

By substitution

$$\xi_1 = u_1 + v, \quad \xi_2 = u_1 - v, \tag{7.114}$$

another canonical form is obtained

$$\frac{\partial^2 u}{\partial u_1^2} - \frac{\partial^2 u}{\partial v^2} = F_1 \equiv 4 \cdot F. \tag{7.115}$$

Reduction of parabolic equation to canonical form

In this case $D = 0$, and we obtain only one real solution of the characteristic equation

$$\xi_1(x_1, x_2) = C_1, \quad \text{where } C_1 \quad \text{is an arbitrary constant,} \tag{7.116}$$

and thus chose transformations in the form

$$\xi_1 = \xi_1(x_1, x_2), \quad \xi_2 = \varphi(x_1, x_2), \tag{7.117}$$

where φ is an arbitrary function, independent of ξ_1.

In this case, we have

$$\bar{a}_{11} = \bar{a}_{12} = 0, \quad \bar{a}_{22} \neq 0, \tag{7.118}$$

and obtain the **canonical form** of the equation of parabolic type

$$\frac{\partial^2 u}{\partial \xi_2^2} = F = -\frac{\bar{b}}{\bar{a}_{22}}. \tag{7.119}$$

Reduction of elliptic equation to canonical form

Given that $D < 0$, the solutions of the characteristic equation are conjugate complex, which yields

$$\xi_1 = C_1, \quad \xi_2 \equiv \xi_1^* = C_2, \tag{7.120}$$

where ξ_1 and ξ_1^* are conjugate complex functions, i.e.

$$\xi_1 = v + iw, \quad \xi_1^* = v - iw. \tag{7.121}$$

As, in this case

$$\bar{a}_{11} = \bar{a}_{22}, \quad \bar{a}_{12} = 0, \tag{7.122}$$

for the **canonical form** we obtain

$$\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2} = F = -\frac{\bar{b}}{\bar{a}_{22}}. \tag{7.123}$$

Canonical form of second order LE with constant coefficients

Observe the equation of the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f(x,y) = 0, \tag{7.124}$$

where a_{11}, a_{12}, a_{22}, b_1, b_2 and c are constants.

As shown before, this equation ca be transformed in one of the following forms

elliptic type

$$u_{\xi\xi} + u_{\eta\eta} + b_1u_\xi + b_2u_\eta + cu + f = 0, \tag{7.125}$$

hyperbolic type

$$u_{\xi\xi} - u_{\eta\eta} + b_1u_\xi + b_2u_\eta + cu + f = 0, \tag{7.126}$$

or

$$u_{\xi\eta} + b_1u_\xi + b_2u_\eta + cu + f = 0, \tag{7.127}$$

parabolic type

$$u_{\xi\xi}+b_1u_\xi+b_2u_\eta+cu+f=0. \tag{7.128}$$

For further simplifications, let us introduce a new function v, instead of u, defined by the following relation

$$u=e^{\lambda\xi+\mu\eta}\cdot v, \tag{7.129}$$

where λ and μ are undefined constants, which will later be chosen to make the transformed form as simple as possible.

From (7.129) we obtain the following relations

$$\frac{\partial u}{\partial\xi}\equiv u_\xi=e^{\lambda\xi+\mu\eta}\cdot\left(v_\xi+\lambda\cdot v\right), \tag{7.130}$$
$$u_\eta=e^{\lambda\xi+\mu\eta}\cdot\left(v_\eta+\mu\cdot v\right), \tag{7.131}$$
$$u_{\xi\xi}=e^{\lambda\xi+\mu\eta}\cdot\left(v_{\xi\xi}+2\lambda\cdot v_\xi+\lambda^2\cdot v\right), \tag{7.132}$$
$$u_{\xi\eta}=e^{\lambda\xi+\mu\eta}\cdot\left(v_{\xi\eta}+\lambda\cdot v_\eta+\mu\cdot v_\xi+\lambda\mu\cdot v\right), \tag{7.133}$$
$$u_{\eta\eta}=e^{\lambda\xi+\mu\eta}\cdot\left(v_{\eta\eta}+2\mu\cdot v_\eta+\mu^2\cdot v\right), \tag{7.134}$$

and thus, for equations of elliptic type, we further obtain

$$\begin{aligned}v_{\xi\xi}+v_{\eta\eta}+(b_1+2\lambda)\,v_\xi+(b_2+2\mu)\,v_\eta+&\\+\left(\lambda^2+\mu^2+b_1\lambda+b_2\mu+c\right)v+f_1=0.&\end{aligned} \tag{7.135}$$

Let us now determine the coefficients λ and μ so that expressions in the first two brackets are annulled

$$\lambda=-\frac{1}{2}b_1\quad \mu=-\frac{1}{2}b_2, \tag{7.136}$$

which thus, according to (7.135), yields

$$v_{\xi\xi}+v_{\eta\eta}+\gamma\cdot v+f_1=0 \tag{7.137}$$

for the elliptic type.

In the previous relation we have introduced the following notation

$$\gamma=\lambda^2+\mu^2+b_1\lambda+b_2\mu+c,\quad f_1=f\cdot e^{-\lambda\xi-\mu\eta}. \tag{7.138}$$

Similarly, for the remaining two cases we obtain

- for the hyperbolic type

$$v_{\xi\eta}+\gamma\cdot v+f_1=0, \tag{7.139}$$

or

$$v_{\xi\xi}-v_{\eta\eta}+\gamma\cdot v+f_1=0, \tag{7.140}$$

- for the parabolic type

$$v_{\xi\xi}+b_2\cdot v_\eta+f_1=0. \tag{7.141}$$

Classification of second order LPE with n variables

It has been previously shown how the classification of second order linear partial equations with two variables is done. In this section, we will generalize this procedure to n variables.

Observe the quadratic form

$$\Phi \equiv \sum_{i,j=1}^{n} a_{ij}^{\circ} y_i y_j, \tag{7.142}$$

where a_{ij}° are constant coefficients, which correspond to coefficients a_{ij} from differential equation (7.81), at point $M_0(x_1^{\circ}, \ldots, x_n^{\circ})$.

It is shown in linear algebra that, for the quadratic form (7.142), a linear transformation can always be chosen

$$y_i = \sum_{k=1}^{n} \alpha_{ik} \eta_k, \tag{7.143}$$

where α_{ik} are real numbers, so that the quadratic form is reduced to a canonical form[10]

$$\Phi = \sum_{i=1}^{n} A_i \eta_i^2, \tag{7.144}$$

where A_i are real numbers.

The differential equation (7.81) at point M_0 is called:

1° an equation of **elliptic type**, if all coefficients A_i are of the same sign;

2° an equation of **hyperbolic type** or **normal-hyperbolic type**, if $n-1$ coefficients A_i are of the same sign, while one is of the opposite sign;

3° an equation of **ultra hyperbolic type**, if m coefficients A_i are of the same sign, while $n-m$ are of the opposite sign, for $m > 1$ and $n-m > 1$;

4° an equation of **parabolic type**, if at least on of he coefficients A_i is equal to zero.

7.4.4 Examples of classification of some equations of mathematical physics

Let us now present examples of the most commonly used partial differential equations.

Example 238

Equation of oscillation in the plane (wave equation)

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + \lambda^2 u = 0, \tag{7.145}$$

that is, in the space

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \lambda^2 u = 0 \tag{7.146}$$

is, according to 2° and $u = u(x_1, x_2, x_3)$, of hyperbolic type.

[10] Canonical or diagonal form. The latter term is more common in linear algebra.

Example 239

Heat conduction equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=a^2\frac{\partial u}{\partial t} \tag{7.147}$$

is, according to 4° and $u=u(x_1,x_2,x_3,t)$, of parabolic type.

Example 240

Laplace equation

$$\Delta u\equiv\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=0 \tag{7.148}$$

is, according to 1° and $u=u(x_1,x_2,x_3)$, of elliptic type.

7.5 A formal procedure for solving LDE

Observe the equation of the form

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=0. \tag{7.149}$$

Further, assume that there exists a solution of the form

$$u=C\cdot e^{\alpha x+\beta y}, \tag{7.150}$$

where α and β are, for now, undetermined constants.

Given that, by assumption, u is a solution, this function must identically satisfy the initial equation, and thus, by differentiating (7.150), followed by substitution of obtained derivatives into (7.149) and dividing by u, we obtain

$$a_{11}\alpha^2+2a_{12}\alpha\cdot\beta+a_{22}\beta^2+2b_1\alpha+2b_2\beta+c=0. \tag{7.151}$$

As we have two arbitrary constants, let us assume that α is an integer, i.e.

$$\alpha=k\quad(k=0,1,\dots). \tag{7.152}$$

From (7.151) we now obtain for β:

$$\beta=ak+b\pm\sqrt{ck^2+dk+f}, \tag{7.153}$$

where a, b, c, d and f are constants that depend of a_{ij}, b_i and c. Given that k is an integer, the value under the square root is not equal to zero, and we obtain, for each k, two solutions

$$C_k\cdot e^{kx+(ak+b+\sqrt{ck^2+dk+f})y},\quad D_k\cdot e^{kx+(ak+b-\sqrt{ck^2+dk+f})y}. \tag{7.154}$$

If C_k and D_k are constants, for the solution we obtain

$$u=\sum_{k=0}^{\infty}C_k\cdot e^{kx+(ak+b+\sqrt{ck^2+dk+f})y}+\sum_{k=0}^{\infty}D_k\cdot e^{kx+(ak+b-\sqrt{ck^2+dk+f})y}. \tag{7.155}$$

R Note that this sum should be taken symbolically, because the convergence of the series that appear here has not been tested.

Depending on the value under the square root, there are two possible cases

$$d^2 = 4cf; \tag{7.156}$$

$$d^2 \neq 4cf. \tag{7.157}$$

In the first case, the value under the square root is a full square

$$ck^2 + dk + f = (mk + n)^2, \tag{7.158}$$

which yields

$$u = \sum_{k=0}^{\infty} C_k \cdot e^{[kx+(a+m)ky+(b+n)y]} + \sum_{k=0}^{\infty} D_k \cdot e^{[kx+(a-m)ky+(b-n)y]}, \tag{7.159}$$

that is

$$u = e^{(b+n)y} \cdot \sum_{k=0}^{\infty} C_k \cdot e^{[kx+(a+m)ky]} + e^{(b-n)y} \cdot \sum_{k=0}^{\infty} D_k \cdot e^{[kx+(a-m)ky]}. \tag{7.160}$$

7.6 The variable separation method

The variable separation method (Fourier method) is one of the most commonly used methods for solving partial differential equation that satisfy given initial and/or boundary (contour) conditions.

The Fourier method can be applied to equations of the form

$$a_{11}u_{xx} + a_{22}u_{yy} + b_1 u_x + b_2 u_y + [F(x) + G(y)]u = 0, \tag{7.161}$$

with initial conditions

$$u(x,0) = f(x), \quad u_y(x,0) = g(x) \tag{7.162}$$

and boundary conditions

$$au(0,y) + bu_x(0,y) = 0, \quad cu(l,y) + du_x(l,y) = 0, \tag{7.163}$$

where f and g are given functions, and a, b, c and d known constants.

Assume that the solution has the form

$$u(x,y) = X(x) \cdot Y(y). \tag{7.164}$$

Let us now find the corresponding derivatives

$$u_x = X' \cdot Y, \quad u_y = Y' \cdot X, \tag{7.165}$$

$$u_{xx} = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} \cdot Y(y) \equiv X'' \cdot Y, \quad u_{yy} = \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} \cdot X(x) \equiv Y'' \cdot X \tag{7.166}$$

and substitute then into the initial equation (7.161)

$$a_{11} \cdot X'' \cdot Y + a_{22} \cdot X \cdot Y'' + b_1 \cdot X' \cdot Y + b_2 \cdot X \cdot Y' + [F(x) + G(y)]XY = 0.$$

From here, dividing by XY, we obtain

$$a_{11}\frac{X''}{X} + a_{22}\frac{Y''}{Y} + b_1\frac{X'}{X} + b_2\frac{Y'}{Y} + F(x) + G(y) = 0, \tag{7.167}$$

that is

$$a_{11}\frac{X''}{X}+b_1\frac{X'}{X}+F(x)=-a_{22}\frac{Y''}{Y}-b_2\frac{Y'}{Y}-G(y). \tag{7.168}$$

As the left hand side of the equation (7.168) is a function only of x, and the right hand side a function only of y, it means that these expressions are constants, namely

$$a_{11}\frac{X''}{X}+b_1\frac{X'}{X}+F(x)=-a_{22}\frac{Y''}{Y}-b_2\frac{Y'}{Y}-G(y)=-\lambda=const. \tag{7.169}$$

In addition to this, the initial conditions must also be satisfied

$$u(x,0)=X(x)\cdot Y(0)=f(x),\quad u_y(x,0)=X(x)\cdot Y'(0)=g(x) \tag{7.170}$$

as well as boundary conditions

$$aX(0)Y(y)+bX'(0)Y(y)=0 \quad\Rightarrow\quad aX(0)+bX'(0)=0, \tag{7.171}$$
$$cX(l)Y(y)+dX'(l)Y(y)=0 \quad\Rightarrow\quad cX(l)+dX'(l)=0. \tag{7.172}$$

The given equation is now decomposed into two ordinary second order differential equations

$$a_{11}X''+b_1X'+F(x)X=-\lambda X \quad\Rightarrow$$
$$a_{11}X''+b_1X'+(F+\lambda)X=0, \tag{7.173}$$
$$a_{22}Y''+b_2Y'+G(y)Y=+\lambda Y \quad\Rightarrow$$
$$a_{22}Y''+b_2Y'+(G-\lambda)Y=0. \tag{7.174}$$

It can be shown, by applying the Sturm – Liouville theory, that there is an infinite number of the so-called **eigenvalues** $\lambda_1,\lambda_2,\ldots$, for which there exist nontrivial solutions (trivial solutions would be $X\equiv 0, Y\equiv 0$) of the equations (7.173) and (7.174). Let X_n ($n=1,2,\ldots$) be the solutions of equation (7.173), for $\lambda=\lambda_n$. The solution of equation (7.174) can be represented in the form

$$Y_n(y)=A_n\overline{Y}_n(y)+B_n\overline{\overline{Y}}_n(y), \tag{7.175}$$

where A_n and B_n are arbitrary constants, and $\overline{Y}_n$ and $\overline{\overline{Y}}_n$ linearly independent particular solutions of equation (7.174), for $\lambda=\lambda_n$. These function are determined from the condition

$$\overline{Y}_n(0)=1;\quad \overline{Y}'_n(0)=0;\quad \overline{\overline{Y}}_n(0)=0;\quad \overline{\overline{Y}}'_n(0)=1. \tag{7.176}$$

Applying the superposition principle , we obtain

$$u(x,y)=\sum_{n=1}^{\infty}X_n(x)\left[A_n\overline{Y}_n(y)+B_n\overline{\overline{Y}}_n(y)\right]. \tag{7.177}$$

This solution must additionally satisfy the conditions

$$\sum_{n=1}^{\infty}A_nX_n(x)=f(x)\qquad \sum_{n=1}^{\infty}B_nX_n(x)=g(x), \tag{7.178}$$

by which the problem is reduced to expanding functions f and g into series of eigenfunctions X_n.

Solving equations of hyperbolic and parabolic type by Fourier method

Solving equations using the Fourier method will be demonstrated on some examples.

Example 241

Find the solution of the equation

$$u_{tt} = a^2 u_{xx}, \quad 0 \le x \le l, \tag{7.179}$$

that satisfies the boundary

$$u(0,t) = 0, \quad u(l,t) = 0, \tag{7.180}$$

and initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{7.181}$$

Solution

Note that, according to classification (under 2°, p. 362), this equation is of hyperbolic type.

According to the Fourier method, let us look for a solution in the form

$$u(x,t) = X(x)\cdot T(t). \tag{7.182}$$

Now we can write the given equation (7.179) in the following form

$$X''\cdot T = \frac{1}{a^2} T''\cdot X \quad \Rightarrow \quad \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda, \tag{7.183}$$

from where it follows

$$X'' + \lambda\cdot X = 0 \quad \wedge \quad T'' + a^2\cdot\lambda T = 0, \tag{7.184}$$

($X(x) \not\equiv 0 \wedge T(t) \not\equiv 0$, as the solutions are not trivial).

The boundary conditions (7.180) come down to

$$u(0,t) = X(0)\cdot T(t) = 0 \quad \Rightarrow \quad \underline{X(0) = 0}, \tag{7.185}$$

$$u(l,t) = X(l)\cdot T(t) = 0 \quad \Rightarrow \quad \underline{X(l) = 0}. \tag{7.186}$$

In this way, looking for the function $X(x)$ is reduced to the problem of **eigenvalues**: find the values λ for which we obtain nontrivial solutions for the problem

$$X'' + \lambda\cdot X = 0, \quad X(0) = X(l) = 0, \tag{7.187}$$

as well as the corresponding solutions.

The values λ obtained in this way are called **eigenvalues**, and the solutions $X(x)$ – **eigenfunctions**. This is the Sturm – Liouville task.

Discussion.

λ can be negative ($\lambda < 0$), zero ($\lambda = 0$) or positive ($\lambda > 0$). Let us observe these three cases:

1° $\lambda < 0$

$$
\begin{gathered}
X(x) = C_1 e^{\sqrt{-\lambda}\cdot x} + C_2 e^{-\sqrt{-\lambda}\cdot x}, \qquad (7.188)\\
X(0) = C_1 + C_2 = 0, \quad X(l) = C_1 e^{\alpha} + C_2 e^{-\alpha} = 0,\\
(\alpha = l\sqrt{-\lambda}) \quad \Rightarrow\\
C_1 = -C_2 \quad \text{i} \quad C_1\left(e^{\alpha} - e^{-\alpha}\right) = 0 \quad \Rightarrow\\
C_1 = C_2 = 0 \quad \Rightarrow \quad X(x) \equiv 0. \qquad (7.189)
\end{gathered}
$$

Thus, in this case, we have only the trivial solution. As we are not interested in this solution, we will now observe the remaining cases.

2° $\lambda = 0$

$$
\begin{gathered}
X(x) = C_1 x + C_2, \qquad (7.190)\\
X(0) = (C_1 x + C_2)|_{x=0} = C_2 = 0,\\
X(l) = C_1 l = 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow\\
X(x) \equiv 0. \qquad (7.191)
\end{gathered}
$$

Thus, in this case, we also have only the trivial solution.

3° $\lambda > 0$

$$
\begin{gathered}
X(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x, \qquad (7.192)\\
X(0) = C_1 = 0,\\
X(l) = C_2 \sin\sqrt{\lambda}\, l = 0 \quad \Rightarrow \qquad (7.193)
\end{gathered}
$$

if $C_2 \neq 0$ (nontrivial solution), then the following must be true

$$
\sin\sqrt{\lambda}\, l = 0 \quad \Rightarrow \quad \sqrt{\lambda} = \frac{\pi n}{l} = \sqrt{\lambda_n}, \qquad (7.194)
$$

and the **nontrivial solution** has the form

$$
X_n(x) = C_n \cdot \sin\frac{\pi n}{l} x. \qquad (7.195)
$$

For $T(t)$ we now obtain $\left(\text{for } \lambda = \lambda_n = \left(\frac{\pi n}{l}\right)^2\right)$

$$
T_n(t) = A_n \cos\left(\frac{\pi n}{l} at\right) + B_n \sin\left(\frac{\pi n}{l} at\right). \qquad (7.196)
$$

Thus, the nontrivial solution of our problem is the function

$$
\begin{aligned}
u_n(x,t) &= X_n(x)\cdot T_n(t) = \qquad (7.197)\\
&= \left[A_n \cos\left(\frac{\pi n}{l} at\right) + B_n \sin\left(\frac{\pi n}{l} at\right)\right] \sin\frac{\pi n}{l} x.
\end{aligned}
$$

According to the superposition principle, a solution is also the function

$$
\begin{aligned}
u &= \sum_{n=1}^{\infty} u_n(x,t) = \qquad (7.198)\\
&= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\pi n}{l} at\right) + B_n \sin\left(\frac{\pi n}{l} at\right)\right] \sin\frac{\pi n}{l} x.
\end{aligned}
$$

The constants are determined from the initial conditions

$$u(x,0)=\varphi(x)=\sum_{n=1}^{\infty}u_n(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{\pi n}{l}x, \tag{7.199}$$

$$u_t(x,0)=\psi(x)=\sum_{n=1}^{\infty}\frac{\partial u_n}{\partial t}\bigg|_{t=0}=\sum_{n=1}^{\infty}\frac{\pi n}{l}aB_n\sin\frac{\pi n}{l}x. \tag{7.200}$$

Thus, the problem is reduced to expanding the known functions $\varphi(x)$ and $\psi(x)$ into Fourier series.

Example 242

Find the solution of the equation

$$a^2u_{xx}=u_t, \quad 0\le x\le l, \quad 0\le t\le t_0, \tag{7.201}$$

that satisfies the initial

$$u(x,0)=\varphi(x), \quad 0\le x\le l, \tag{7.202}$$

and boundary conditions

$$u(0,t)=0, \quad u(l,t)=0 \quad 0\le t\le t_0. \tag{7.203}$$

Solution

Let us first note that, according to 4° (p. 362), this equation is of parabolic type.

We shall look for a solution in the form

$$u(x,t)=X(x)\cdot T(t) \quad \Rightarrow \tag{7.204}$$

$$u_t=X\cdot\dot{T}, \quad u_x=X'\cdot T, \quad u_{xx}=X''\cdot T, \tag{7.205}$$

where $X\not\equiv 0$ and $T\not\equiv 0$, as we are looking for a nontrivial solution. Substituting (7.204) into equation (7.201) we obtain

$$a^2X''T=X\dot{T}, \quad \text{that is}$$

$$\frac{X''}{X}=\frac{1}{a^2}\frac{\dot{T}}{T}=-\lambda. \tag{7.206}$$

Thus, the initial equation (7.201) is now decomposed to two ordinary differential equations:

$$X''+\lambda X=0 \quad \wedge \quad \dot{T}+a^2\lambda T=0, \tag{7.207}$$

from where we obtain, as in the previous case, for $X(x)$

$$X_n(x)=\bar{C}_n\sin\frac{\pi n}{l}x, \tag{7.208}$$

while for $T(t)$ we obtain

$$T_n(t)=\bar{\bar{C}}_ne^{-a^2\lambda_nt}, \tag{7.209}$$

where $\bar{\bar{C}}_n$ is, for now, undetermined.

Thus, for u_n we obtain

$$u_n = C_n e^{-a^2\lambda_n t} \sin\frac{\pi n}{l}x, \quad (C_n = \bar{C}_n \bar{\bar{C}}_n). \tag{7.210}$$

Finally, according to the superposition principle, for $u(x,t)$ we obtain

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} C_n e^{-a^2\lambda_n t} \sin\frac{\pi n}{l}x. \tag{7.211}$$

Initial conditions (7.202) are used for determining the constants C_n, which yields

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} C_n \sin\frac{\pi n}{l}x. \tag{7.212}$$

Once again, we have reduced the problem to expansion of known functions into Fourier series

$$C_n = \frac{2}{l}\int_0^l \varphi(\xi)\sin\left(\frac{\pi n}{l}\xi\right)\,\mathrm{d}\xi. \tag{7.213}$$

We have thus solved the given task.

Solving equations of elliptic type by Fourier method

Example 243

Observe the Laplace equation in spherical coordinates, where $u = u(r,\varphi,\theta)$

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{7.214}$$

In the case of spherical symmetry, i.e. if $u = u(r)$, we have

$$\frac{\partial u}{\partial\theta} = 0, \quad \frac{\partial u}{\partial\varphi} = 0 \quad \text{i} \quad \frac{\partial u}{\partial r} = \frac{\mathrm{d}u}{\mathrm{d}r}, \tag{7.215}$$

and the Laplace equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u}{\mathrm{d}r}\right) = 0 \quad \Rightarrow \tag{7.216}$$

$$u = -\frac{C_1}{r} + C_2, \tag{7.217}$$

where C_1 and C_2 are arbitrary constants. Assume, for example, that $C_1 = -1$, and $C_2 = 0$, which yields

$$u_o = \frac{1}{r} \tag{7.218}$$

the **basic solution of the Laplace equation in the space**.

R Note that the function u_o satisfies the Laplace equation everywhere, except at point $r = 0$.

Example 244

Observe now the Laplace equation in cylindrical coordinates $u = u(\rho, \varphi, z)$

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{7.219}$$

In the case that u does not depend of φ and z, i.e. $u = u(\rho)$, we obtain

$$\Delta u = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}u}{\mathrm{d}\rho}\right) = 0, \tag{7.220}$$

and it follows that

$$\rho\frac{\mathrm{d}u}{\mathrm{d}\rho} = C_1 \quad \Rightarrow \quad \mathrm{d}u = \frac{C_1}{\rho}\mathrm{d}\rho \quad \Rightarrow \quad u = C_1 \ln\rho + C_2. \tag{7.221}$$

The constants C_1 and C_2 are arbitrary, and if we select $C_1 = -1$ and $C_2 = 0$, we obtain:

$$u_o = u_o(\rho) = \ln\frac{1}{\rho}. \tag{7.222}$$

This function is often also called the **basic solution of the Laplace equation in the plane** (for two independent variables). The function u_o satisfies the Laplace equation everywhere (in the plane), except at point $\rho = 0$.

7.7 Green formulas

Let us start from the Ostrogradsky formula (4.92):

$$\iiint\limits_V \mathrm{div}\,\mathbf{a}\,\mathrm{d}V = \iint\limits_S \mathbf{a}\,\mathrm{d}\mathbf{S} = \iint\limits_S a_n\,\mathrm{d}S, \tag{7.223}$$

where

$$\begin{aligned} \mathrm{div}\,\mathbf{a} &= \mathrm{div}(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}), \\ \mathbf{a}\cdot\mathbf{n} &= a_n, \quad \mathrm{d}\mathbf{S} = dS\cdot\mathbf{n}, \\ a_n &= a_x\cdot\cos\alpha + a_y\cdot\cos\beta + a_z\cdot\cos\gamma, \\ \alpha = \angle(\mathbf{n},\mathbf{i}), &\quad \beta = \angle(\mathbf{n},\mathbf{j}), \quad \gamma = \angle(\mathbf{n},\mathbf{k}), \end{aligned} \tag{7.224}$$

$\mathbf{n}$ is the outer normal of the closed surface S, which limits the space V, and a_x, a_y and a_z are arbitrary differentiable functions. Here, the indices x, y, z denote the projections of the corresponding values on the x, y and z axes, respectively. Not to be identified with derivatives (in this case)!!!

Let us now introduce new scalar functions $u(x,y,z)$ and $v(x,y,z)$, which are continuous, and whose first derivatives on the border S, and second derivatives inside the region V, are continuous.

Let us subsequently introduce

$$a_x = u\cdot\frac{\partial v}{\partial x}, \quad a_y = u\cdot\frac{\partial v}{\partial y}, \quad a_z = u\cdot\frac{\partial v}{\partial z}, \quad \text{i.e.} \quad \mathbf{a} = u\cdot\mathrm{grad}\,v, \tag{7.225}$$

and then the relation (7.223) can be written in the form

$$\iiint\limits_V \operatorname{div}(u\cdot \operatorname{grad} v)\,\mathrm{d}V = \iint\limits_S u\cdot \operatorname{grad} v\cdot \mathrm{d}\mathbf{S} \quad \Rightarrow$$

$$\iiint\limits_V (\operatorname{grad} u\cdot \operatorname{grad} v + u\cdot \operatorname{div}\operatorname{grad} v)\,\mathrm{d}V = \iint\limits_S u\cdot \operatorname{grad} v\cdot \mathbf{n}\cdot \mathrm{d}S \quad \Rightarrow$$

$$\iiint\limits_V u\cdot \Delta v\,\mathrm{d}V = \iint\limits_S u\cdot \operatorname{grad} v\cdot \mathbf{n}\cdot \mathrm{d}S - \iiint\limits_V \operatorname{grad} u\cdot \operatorname{grad} v\cdot \mathrm{d}V \tag{7.226}$$

From here follows the **first Green formula**

$$\iiint\limits_V u\cdot \Delta v\,\mathrm{d}V = \iint\limits_S u\cdot \frac{\partial v}{\partial n}\,\mathrm{d}S - \iiint\limits_V \nabla u\cdot \nabla v\,\mathrm{d}V. \tag{7.227}$$

Assume now that $\mathbf{a} = v\cdot \operatorname{grad} u$, and we obtain similarly

$$\iiint\limits_V v\cdot \Delta u\,\mathrm{d}V = \iint\limits_S v\cdot \frac{\partial u}{\partial n}\,\mathrm{d}S - \iiint\limits_V \nabla v\cdot \nabla u\,\mathrm{d}V. \tag{7.228}$$

Subtracting equation (7.228) from (7.227) we obtain

$$\iiint\limits_V (u\cdot \Delta v - v\cdot \Delta u)\,\mathrm{d}V = \iint\limits_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S, \tag{7.229}$$

the **second Green formula**.

Note that the relation

$$\Delta u = f(x,y,z) \tag{7.230}$$

is called the **Poisson equation**, where u is the unknown function, while f is known.

Equation of the form

$$\Delta u = 0 \tag{7.231}$$

as already mentioned, is called the Laplace equation. Each continuous function $u = u(x,y,z)$, which satisfies the Laplace equation (7.231), is called a **harmonic function**. Note that it is assumed that its first and second derivatives (which appear in the expression (7.231)) are also continuous functions.

Let us now state some theorems, which we will use later in our proofs:

Theorem 25

If S is a closed surface and u and v are harmonic functions, then

$$\iint\limits_S (u\cdot \operatorname{grad} \mathrm{v})\,\mathrm{d}\mathbf{S} = \iint\limits_S (\mathrm{v}\cdot \operatorname{grad} \mathrm{u})\,\mathrm{d}\mathbf{S}, \tag{7.232}$$

or, written differently

$$\iint\limits_S u\cdot \frac{\partial v}{\partial n}\,\mathrm{dS} = \iint\limits_S \mathrm{v}\cdot \frac{\partial \mathrm{u}}{\partial \mathrm{n}}\,\mathrm{dS}. \tag{7.233}$$

Proof

By assumption, u and v are harmonic functions, i.e. $\Delta u = 0$, $\Delta v = 0$, and thus from (7.229) the statement (7.233) follows directly.

Theorem 26

If S is a closed surface, which limits a part of the space V, and U a harmonic function, then

$$\iiint_V \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right] \mathrm{dV} = \iint_S \mathrm{U} \cdot \frac{\partial \mathrm{U}}{\partial \mathrm{n}} \mathrm{dS}. \tag{7.234}$$

Proof

Let us now apply the second Green formula (7.229), assuming $u = U^2$ and $v \equiv 1$. We will first calculate the Laplacian

$$\frac{\partial u}{\partial x} = 2U \frac{\partial U}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2\left(\frac{\partial U}{\partial x}\right)^2 + 2 \cdot U \frac{\partial^2 U}{\partial x^2}. \tag{7.235}$$

Similarly, we obtain also

$$\frac{\partial^2 u}{\partial y^2} = 2\left(\frac{\partial U}{\partial y}\right)^2 + 2 \cdot U \frac{\partial^2 U}{\partial y^2}, \quad \frac{\partial^2 u}{\partial z^2} = 2\left(\frac{\partial U}{\partial z}\right)^2 + 2 \cdot U \frac{\partial^2 U}{\partial z^2}, \tag{7.236}$$

and thus

$$\Delta u = 2U \cdot \Delta U + 2\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]. \tag{7.237}$$

If we now use the assumption that U is a harmonic function, i.e. $\Delta U = 0$, then from (7.237), for Δu, we obtain

$$\Delta u = 2\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]. \tag{7.238}$$

Finally, by including (7.238) and $v = 1$ into (7.229), we obtain

$$\iiint_V \left[\frac{\partial U}{\partial x}^2 + \frac{\partial U}{\partial y}^2 + \frac{\partial U}{\partial z}^2\right] \mathrm{d}V = \iint_S U \frac{\partial U}{\partial n} \mathrm{d}S, \tag{7.239}$$

which was to be proved.

When solving partial differential equations, it can also be required that the solution satisfies specific, the so-called **contour conditions**[11], when the values of a function on the surface S (contour - boundary) are known, and we are looking for them in the interior of the region V. Depending on these conditions, we distinguish the following problems:

[11] In literature, these conditions are also often called "boundary conditions".

First contour problem or Dirichlet[12] problem.

If the value of the function $u(x,y,z)$ on the border S is known, i.e. $u(x,y,z) = f_1$ on S, determine its value inside the region V. The function f_1 is known.

Second contour problem or Neumann[13] problem.

If the derivative of the function $u(x,y,z)$ on the surface S is known, i.e. $\dfrac{\partial u}{\partial n} = f_2$ on S, determine the value of the function u. The function f_2 is known.

Third contour problem or mixed contour problem.

This problem is a combination of the previous two. Namely if the values of a function $u(x,y,z)$ and its derivative on the contour S are known, find its value within the region V, i.e.

$$\left.\frac{\partial u}{\partial n}\right|_S + h\,(u - f_3) = 0. \tag{7.240}$$

Dirichlet problem

Find the solution of the Laplace equation $\Delta u = 0$ that satisfies the predefined condition

$$u(x,y,z) = f(x,y,z) \quad \text{on the contour} \quad S, \tag{7.241}$$

or shortly

$$u|_S = f. \tag{7.242}$$

In the case of two variables, say x, y, S is a simple closed curve, without singularities.

R Note that this problem can have only one solution.

In order to solve this problem, observe an arbitrary point $A(a,b,c)$ of the region V, which belongs to a sub-region Σ

$$\sum = \{x,y,z| \quad (x-a)^2 + (y-b)^2 + (z-c)^2 \le R^2\}. \tag{7.243}$$

R is determined in such a way that $\Sigma \subset V$. Let us denote the border of the region Σ as σ.

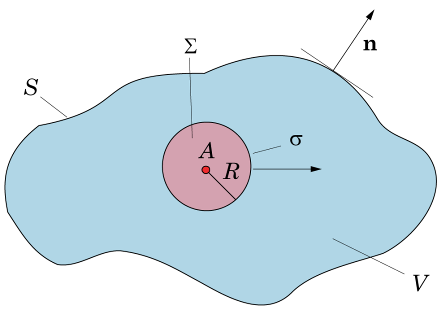

Figure 7.3: Contour conditions.

[12]Peter Gustav Lejeune-Dirichlet (1805-1859), German mathematician. Author of significant papers in analysis, number theory and algebraic structures. He proved the convergence of Fourier series and formulated the general conditions under which a function can be expressed in the form of a trigonometric series.

[13]Carl Gottfried Neumann (1832-1925), German mathematician. Known for defining this problem.

Let u and v be two **harmonic functions** in region V. According to the second Green formula (7.229) and Theorem 25 (p. 371) (7.233), we obtain

$$\iint_{S\cup\sigma} u\frac{\partial v}{\partial n}\,\mathrm{d}S - \iint_{S\cup\sigma} v\frac{\partial u}{\partial n}\,\mathrm{d}S = 0. \tag{7.244}$$

We have seem that the function

$$\frac{1}{r} = \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} \tag{7.245}$$

is harmonic, so let us assume that $v = 1/r$. For such a choice of v the relation (7.244) becomes

$$\iint_{S\cup\sigma} \left(-u\frac{1}{r^2} - \frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0, \tag{7.246}$$

that is

$$\iint_{S} \left(-u\frac{1}{r^2} - \frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S + \iint_{\sigma} \left(-u\frac{1}{r^2} - \frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0. \tag{7.247}$$

Given that σ is a sphere, with boundary equation $r = R$, we obtain

$$\iint_{S} \left(-u\frac{1}{r^2} - \frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S = \frac{1}{R^2}\iint_{\sigma} u\,\mathrm{d}S + \frac{1}{R}\iint_{\sigma}\frac{\partial u}{\partial n}\,\mathrm{d}S. \tag{7.248}$$

Integrals at the right hand side of the previous equation can be represented as follows

$$\frac{1}{R^2}\iint_{\sigma} u\,\mathrm{d}S = \frac{1}{R^2}4\pi R^2 u^* = 4\pi u^*, \tag{7.249}$$

where u^* is the mean value of function u on surface σ, (it is known, does not depend of S and can be moved in front of the integral), ($4\pi R^2$ is the area of σ, i.e, the area of the sphere) and

$$\frac{1}{R}\iint_{\sigma}\frac{\partial u}{\partial n}\,\mathrm{d}S = \frac{1}{R}4\pi R^2\left(\frac{\partial u}{\partial n}\right)^*, \tag{7.250}$$

where $\left(\frac{\partial u}{\partial n}\right)^*$ is the mean value of the derivative on the sphere.

Further, given that

$$\lim_{R\to 0} u^* = u(A), \quad \lim_{R\to 0} 4\pi R\left(\frac{\partial u}{\partial n}\right)^* = 0, \tag{7.251}$$

$$\left[\iint_{\sigma}\mathrm{d}S = 4\pi R^2, \quad \lim_{(x,y,z)\to A} u(x,y,z) = u(A),\right.$$
$$\left.\lim_{(x,y,z)\to A}\iint_{\sigma} u(x,y,z)\,\mathrm{d}S = u(A)\iint_{\sigma}\mathrm{d}S = u(A)4\pi R^2,\right]$$

and that on the surface S: $u|_S = f$, we obtain, from (7.248)

$$4\pi u(A) = -\iint_{S}\left(\frac{f}{r^2} + \frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S. \tag{7.252}$$

Finally, as

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \tag{7.253}$$

we obtain

$$u(a,b,c) = \frac{1}{4\pi}\iint\limits_S \left[f\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] \mathrm{d}S. \tag{7.254}$$

This relation, for now, does not give the solution of the problem, as there exists an unknown value $\partial u/\partial n$ under the integral.

R Note that we could solve the problem in a similar way when the function depends on two variables, x,y. In this case, we would replace the surface with a line l, and also replace $\sigma = 4\pi R^2$ by $l = 2\pi R$.

Let us solve the previous problem (Dirichlet problem) in two special cases, when the region Σ is:

- a circle and
- a sphere.

Dirichlet problem for a circle

The task is to find a function u that satisfies the Laplace equation $\Delta u = 0$ within the circle $K = \left\{(x,y) | x^2 + y^2 = R_0^2\right\}$.

Due to the nature of the problem, it is more convenient tot use polar coordinates (r,φ), in which the Laplace equation takes the form

$$r\frac{\partial u}{\partial r} + r^2\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{7.255}$$

Let us look for a solution in the form

$$u = R(r)\cdot F(\varphi), \tag{7.256}$$

and thus the equation (7.255) becomes

$$\left(r^2R'' + r\cdot R'\right)F + R\cdot F'' = 0, \tag{7.257}$$

that is

$$\frac{r^2\cdot R'' + r\cdot R'}{R} = -\frac{F''}{F} = k^2, \tag{7.258}$$

where k is a constant.

We shall analyze two cases: $k \neq 0$ and $k = 0$.

For $k \neq 0$ we obtain:

$$r^2R'' + r\cdot R' - R\cdot k^2 = 0 \quad \text{- Euler equation and} \tag{7.259}$$

$$F'' + k^2F = 0 \quad \text{- diff. eq. with const. coefficients.} \tag{7.260}$$

The solutions of these equations are

$$R(r) = C_1 r^k + C_2 r^{-k}, \quad F(\varphi) = C_3\cos(k\varphi) + C_4\sin(k\varphi), \tag{7.261}$$

where C_i $(i = 1,2,3,4)$ are arbitrary constants.

For $k=0$ we obtain

$$rR''+R'=0,$$
$$F''=0.$$

with the substitution $R'=f(r)$, for the first equation, we obtain $df/dr=-f/r$ or

$$\ln f=-\ln r+\ln C_1 \quad\Rightarrow\quad rf=C_1=r\frac{\mathrm{d}R}{\mathrm{d}r} \quad\Rightarrow \tag{7.262}$$
$$R=C_1\ln r+C_2. \tag{7.263}$$

For the second equation we obtain $F=C_3\cdot\varphi+C_4$.
Thus, the solution of the initial equation is:
for $k\neq 0$

$$u=\left(C_1r^k+C_2r^{-k}\right)\cdot(C_3\cos(k\varphi)+C_4\sin(k\varphi)) \tag{7.264}$$

for $k=0$

$$u=(C_1\ln r+C_2)\cdot(C_3\varphi+C_4). \tag{7.265}$$

We determine the constants from the condition that the solution satisfies the boundary conditions. Namely, we have assumed that on the circular contour the function u has some predefined value. However, as after completing the round of the circle we reach the same point again, this solution must be periodic with period 2π. This practically means that k must be an integer $\pm1,\pm2,\ldots$, and $C_3=0$ in (7.265). We would obtain the same solutions if we assumed that k is a natural number, i.e. $k=1,2,\ldots,n$, so we finally obtain

$$u_0=u|_{k=0}=C_4\left(C_1\ln r+C_2\right), \tag{7.266}$$
$$u_n=u|_{k=n}=\left(C_nr^n+D_nr^{-n}\right)\left(a_n\cos(n\varphi)+b_n\sin(n\varphi)\right),\quad n=1,2,\ldots \tag{7.267}$$

Further on, the function u must be continuous, by assumption, at all point inside the region bounded by K, and thus at point $r=0$ as well. However, as the functions $\ln r$ and r^{-n} are not defined at that point, it follows that the coefficients next to them are equal to zero ($C_1=D_n=0$), and the solution gains the form

$$u_0(r,\varphi)=\frac{a_0}{2};\quad u_n(r,\varphi)=r^n\left(a_n\cos(n\varphi)+b_n\sin(n\varphi)\right). \tag{7.268}$$

Applying the superposition principle, for u we obtain

$$u(r,\varphi)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}r^n\left(a_n\cos(n\varphi)+b_n\sin(n\varphi)\right). \tag{7.269}$$

We have not yet used the condition at the boundary K.

Let $u(R,\varphi)=u|_{r=R}=f(\varphi)$, where f is a known function. From (7.269) we obtain

$$f(\varphi)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}R^n\left(a_n\cos(n\varphi)+b_n\sin(n\varphi)\right). \tag{7.270}$$

Thus, a known function should be expanded into a Fourier series, which yields

$$a_n=\frac{1}{\pi R^n}\int_{-\pi}^{+\pi}f(t)\cos(nt)\,\mathrm{d}t,\quad b_n=\frac{1}{\pi R^n}\int_{-\pi}^{+\pi}f(t)\sin(nt)\,\mathrm{d}t. \tag{7.271}$$

Substituting these values into (7.269) we obtain

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,\mathrm{d}t+\frac{1}{\pi}\sum_{n=1}^{+\infty}\int_{-\pi}^{+\pi}\frac{r^n}{R^n}f(t)\cos n(t-\varphi)\,\mathrm{d}t, \tag{7.272}$$

or

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[1+2\sum_{n=1}^{+\infty}\left(\frac{r}{R}\right)^n\cos n(t-\varphi)\right]f(t)\,\mathrm{d}t. \tag{7.273}$$

However, as

$$1+2\sum_{n=1}^{\infty}c^n\cos(n\varphi)=\frac{1-c^2}{c^2-2c\cos\varphi+1},\quad 0\le c\le 1, \tag{7.274}$$

we finally obtain the solution of the Dirichlet problem for a circle

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\frac{R^2-r^2}{R^2-2rR\cos(t-\varphi)+r^2}\,\mathrm{d}t. \tag{7.275}$$

This relation is known in literature also as the **Poisson integral**.

Dirichlet problem for a sphere

In this case, the problem is reduced to finding a function u that satisfies the Laplace equation $\Delta u=0$, and whose value on the sphere S is known ($u|_S=f$).

Observe now the sphere S, with radius R, and two points: $A(a,b,c)$ in the interior of the sphere and $A_1(a_1,b_1,c_1)$ out of the sphere.

Let O be the center of the sphere, and point $M(x,y,z)\in S$ (Fig. 5.4).

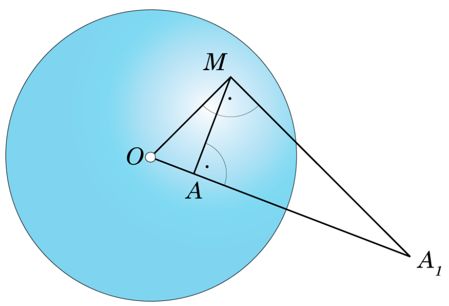

Figure 7.4:

Let us introduce the following notation

$$\overline{AM}=r,\quad \overline{A_1M}=r_1,\quad \overline{OA}=l,\quad \overline{OA_1}=l_1 \tag{7.276}$$

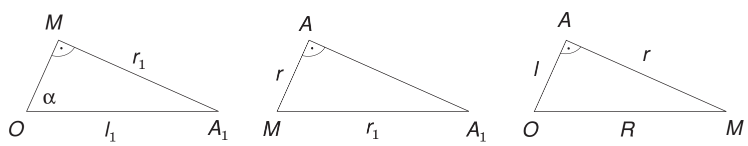

Figure 7.5:

From the similarity of triangles (Fig. 5.5) follows the proportionality of their sides

$$\frac{r}{r_1} = \frac{l}{R} = \frac{R}{l_1} \quad \Rightarrow \quad \frac{1}{r_1} = \frac{l}{R}\frac{1}{r}. \tag{7.277}$$

Given that $1/r$ is a harmonic function in the region bounded by the sphere S, we can apply the equation (7.254)

$$u(a,b,c) = \frac{1}{4\pi}\iint\limits_S \left[f\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] \mathrm{d}S. \tag{7.278}$$

According to (7.277) and the second Green formula (7.229) we get

$$\iint\limits_S \left[u\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right) - \frac{1}{r_1}\frac{\partial u}{\partial n}\right] \mathrm{d}S = 0. \tag{7.279}$$

Further on, according to the assumptions of this problem, we have that $u = f$ for $M(x,y,z) \in S$, and thus we obtain

$$\iint\limits_S \left[f\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right) - \frac{1}{r_1}\frac{\partial u}{\partial n}\right] \mathrm{d}S = 0 \Bigg| \cdot \frac{R}{4\pi l} \quad \Rightarrow$$

$$\frac{R}{4\pi l}\iint\limits_S \left[f\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right) - \frac{1}{r_1}\frac{\partial u}{\partial n}\right] \mathrm{d}S = 0. \tag{7.280}$$

R Note that we could not apply this relation to function $1/r$, as it is not defined at point $A(a,b,c)$, inside the region S, while the function $1/r_1$ is not defined at point $A_1(a_1,b_1,c_1)$, which is outside the observed region.

Finally, from (7.280) and (7.278), and using (7.277), we obtain

$$u(a,b,c) = \frac{1}{4\pi}\iint\limits_S \left[f\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{R}{l} f\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right)\right] \mathrm{d}S \quad \Rightarrow \tag{7.281}$$

$$u(a,b,c) = \frac{1}{4\pi}\iint\limits_S f\frac{\partial}{\partial n}\left(\frac{1}{r} - \frac{R}{l}\frac{1}{r_1}\right) \mathrm{d}S$$

the solution for Dirichlet problem for a sphere.

Let us now define the function G by the relation

$$G = G(x,y,z,a,b,c) = \frac{1}{r} - \frac{R}{l}\frac{1}{r_1} = \tag{7.282}$$

$$= \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} - \frac{R}{l}\frac{1}{\sqrt{(x-a_1)^2+(y-b_1)^2+(z-c_1)^2}}$$

with the following properties:

- G is a harmonic function with respect to point $M(x,y,z)$, within the sphere S, except at point $A(a,b,c)$;
- G is a harmonic function with respect to point $A(a,b,c)$, within the sphere S, except at point $M(x,y,z)$;
- function $G - 1/r$ is a harmonic function at all points points inside the region S;
- G is annulled on the sphere S.

The function G, defined in this way, is called the **Green function** with respect to point $A(a,b,c)$ of the sphere S.

Observe now a known function $H(x,y,z,a,b,c)$, a fixed point $A(a,b,c)$ in region V and an arbitrary point $M(x,y,z)$ in this region (H is a function of A and M).

Assume that this function has the following properties:

1° it is harmonic with respect to point $M(x,y,z)$;
2° it is harmonic with respect to point $A(a,b,c)$;
3° its value on the surface S is $1/r$, where $r = \overline{AM}$.

Let u be a solution of the Dirichlet problem, then, according to (7.229)

$$\iint_S \left(u\frac{\partial H}{\partial n} - H\frac{\partial u}{\partial n} \right) \mathrm{d}S = 0, \tag{7.283}$$

and according to 3° we obtain:

$$\iint_S \left(f\frac{\partial H}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} \right) \mathrm{d}S = 0 \quad \Rightarrow$$

$$\frac{1}{4\pi}\iint_S f\frac{\partial H}{\partial n}\,\mathrm{d}S = \frac{1}{4\pi}\iint_S \frac{1}{r}\frac{\partial u}{\partial n}\,\mathrm{d}S. \tag{7.284}$$

Further, according to (7.254), for a "solution" of the Dirichlet problem we obtain

$$u(a,b,c) = \frac{1}{4\pi}\iint_S \left(f\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) \mathrm{d}S \quad \Rightarrow$$

$$u(a,b,c) = \frac{1}{4\pi}\iint_S \left(f\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) \mathrm{d}S \quad \Rightarrow$$

$$u(a,b,c) = \frac{1}{4\pi}\iint_S f\frac{\partial}{\partial n}\left(\frac{1}{r} - H\right) \mathrm{d}S \tag{7.285}$$

Thus, this relation (7.285) gives a solution of the Dirichlet problem, if the function H is known.

The function defined by

$$G = H - \frac{1}{r} \tag{7.286}$$

is called the **Green function** for region V, with respect to point $A(a,b,c)$. The region is bounded by the surface S.

Based on the definition of this function and previous assumptions, we conclude that:

1° G is a harmonic function in region V, with respect to point $M(x,y,z)$, except at point $A(a,b,c)$;
2° G is a harmonic function in region V, with respect to point $A(a,b,c)$, except at point $M(x,y,z)$;
3° function $G - 1/r$ is harmonic at all points of region V;
4° on the surface S the function G is annulled.

Neumann problem in a plane

Let P be a given region in a plane, bounded by the curve ℓ, while on ℓ a function $f(s)$, $s \in \ell$ is defined.

The Neumann problem consist of the following: find a function u, which is harmonic in region P, and whose derivative in the direction of the normal $\partial u/\partial n$ is a known function f on the contour ℓ, i.e.

$$\frac{\partial u}{\partial n} = f(s), \quad s \in \ell. \tag{7.287}$$

Theorem 27

In order for the solution of the Neumann problem to exist, it is necessary that the integral of the function f is annulled on the contour ℓ.

Proof

Given that u is a harmonic function, it follows that

$$\iint_S \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}x\cdot\mathrm{d}y=0. \tag{7.288}$$

Further, according to the Green formula (7.229), for $v=1$, we obtain, for the case in the plane

$$\iint_S \Delta u\,\mathrm{d}S=\int_\ell \frac{\partial u}{\partial n}\mathrm{d}l=\int_\ell f\,\mathrm{d}l=0. \tag{7.289}$$

Theorem 28

Two solutions of the Neumann problem can differ only by an arbitrary constant.

Proof

Let us prove this Theorem for the case when $f(s)=0$.

Assume that $f(s)=0$. Then $u=0$ is a solution of the respective Neumann problem. Let v be some other solution of the same problem. Given that v is a harmonic function, we obtain

$$\iint_S v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y=0. \tag{7.290}$$

However, as

$$\iint_S v\frac{\partial^2 v}{\partial x^2}\mathrm{d}x\mathrm{d}y=\int_\ell v\frac{\partial v}{\partial x}\mathrm{d}y-\iint_S\left(\frac{\partial v}{\partial x}\right)^2\mathrm{d}x\mathrm{d}y \tag{7.291}$$

and

$$\iint_S v\frac{\partial^2 v}{\partial y^2}\mathrm{d}x\mathrm{d}y=\int_\ell v\frac{\partial v}{\partial y}\mathrm{d}x-\iint_S\left(\frac{\partial v}{\partial y}\right)^2\mathrm{d}x\mathrm{d}y, \tag{7.292}$$

by adding the previous relations we obtain

$$\begin{aligned}\iint_S v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y=\int_\ell v\frac{\partial v}{\partial n}\mathrm{d}s-&\\ -\iint_S\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y.&\end{aligned} \tag{7.293}$$

Given that, according to the assumption, $\Delta v=0$ and $\dfrac{\partial v}{\partial n}=0$, it follows that

$$\iint_S\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y=0, \tag{7.294}$$

from where, due to the continuity of the derivative $\partial v/\partial x$ and $\partial v/\partial y$, it follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = \text{const.} \tag{7.295}$$

This proves the Theorem.

7.8 Examples

Problem 245

Find the general solution of equation

$$\frac{\partial f}{\partial x}(=f_x)=0, \quad \text{where } f=f(x,y).$$

Solution

$$\frac{\partial f}{\partial x}=0 \quad \Rightarrow \quad f=f(y),$$

where f is an arbitrary differentiable function of the variable y.

Problem 246

Find the general solution of equation[14]

$$\frac{\partial^2 f}{\partial x \partial y}=0.$$

Solution

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=0, \\ \text{or} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)=0, \end{array}\right. \quad \Rightarrow$$

$$\left\{\begin{array}{l} \dfrac{\partial f}{\partial y}=\varphi(y) \quad \Rightarrow \quad f=\int \varphi(y)\,\mathrm{d}y+\psi(x), \\ \text{or} \\ \dfrac{\partial f}{\partial x}=\phi(x) \quad \Rightarrow \quad f=\int \phi(x)\,\mathrm{d}x+\gamma(y). \end{array}\right. \quad \Rightarrow$$

Thus, the general solution is a function of the form

$$f=\varphi(x)+\psi(y),$$

where φ and ψ are arbitrary differentiable functions of x and y, respectively.

[14]The given equation is a partial second order equation. However, by a simple substitution ($\partial f/\partial y=\varphi(y)$ or $\partial f/\partial x=\phi(x)$) it is reduced to a first order equation, which is why it is placed here.

Problem 247

Find the general solution of equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0.$$

Solution

To this PDE a system of equations is assigned of the form (see (**??**) on p. **??**)

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}.$$

It has been said that the solution is

$$\psi_i=C_i,\quad i=1,\ldots,n-1,\ \text{where } n \text{ is the number of independent variables,}$$

and ψ_i are first integrals. In this case $n=2$, and we thus have only one first integral

$$\ln y=\ln x+\ln c_1 \quad\Rightarrow\quad \ln y=\ln c_1 x \quad\Rightarrow\quad y=c_1 x \quad\Rightarrow\quad \frac{y}{x}=c_1,\quad c_1>0.$$

Thus, the general solution is of the form

$$f=f\left(\frac{y}{x}\right),$$

where f is an arbitrary differentiable function of the variable y/x.

Problem 248

Find the solution of the partial differential equation

$$(x+1)\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+(z-1)\frac{\partial f}{\partial z}=0.$$

Solution

To the given partial equation we assign the system of equations

$$\frac{\mathrm{d}x}{x+1}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z}{z-1},$$

where $n=3$, so there are $n-1=2$ first integrals. First integrals are (from the first pair of equations)

$$\ln(x+1)+\ln c_1=\ln y \quad\Rightarrow\quad c_1=\frac{y}{x+1} \text{ or } C_1=\frac{x+1}{y},$$

that is (from the second pair of equations)

$$\ln y=\ln(z-1)+\ln c_2 \quad\Rightarrow\quad c_2=\frac{y}{z-1} \text{ or } C_2=\frac{z-1}{y},\quad c_1>0,\, c_2>0.$$

Finally we obtain the general solution

$$f = f\left(\frac{y}{x+1}, \frac{y}{z-1}\right) \text{ or } g = g\left(\frac{x+1}{y}, \frac{z-1}{y}\right),$$

where f and g are arbitrary differentiable functions by corresponding variables.

R Note that the functions f and g are solutions of our problem and any of these two functions is called the general solution.

Problem 249

Find the solution of the differential equation

$$\sqrt{x}\frac{\partial u}{\partial x} + \sqrt{y}\frac{\partial u}{\partial y} + \sqrt{z}\frac{\partial u}{\partial z} = 0,$$

which for $x = 1$ becomes $u = y - z$.

Solution

This is the so-called Cauchy problem. We will first find the general solution. To the initial equation corresponds the system

$$\frac{\mathrm{d}x}{\sqrt{x}} = \frac{\mathrm{d}y}{\sqrt{y}} = \frac{\mathrm{d}z}{\sqrt{z}},$$

which has two first integrals

$$\frac{\mathrm{d}x}{\sqrt{x}} = \frac{\mathrm{d}y}{\sqrt{y}} \quad \Rightarrow \quad \sqrt{x} - \sqrt{y} = c_1 = \psi_1$$

and, similarly, the second first integral

$$\sqrt{x} - \sqrt{z} = c_2 = \psi_2.$$

Let is now find the Cauchy solution (Cauchy integral). According to the theory

$$\psi_i(x_1, \ldots, x_n) = c_i, \quad i = 1, \ldots, n-1$$

$$\left.\begin{array}{l} \psi_1(x_0, y, z) = \bar{\psi}_1 \\ \psi_2(x_0, y, z) = \bar{\psi}_2 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} y = \lambda_2(\bar{\psi}_1, \bar{\psi}_2), \\ z = \lambda_3(\bar{\psi}_1, \bar{\psi}_2) \end{array}$$

is a solution of equation $F[z] = 0$ that satisfies the condition

$$z(x_0, y) = \varphi(y), \quad u = \varphi\left[\lambda_2(\psi_1, \psi_2), \lambda_3(\psi_1, \psi_2)\right]$$

. In our case

$$\begin{aligned} -\sqrt{x} + \sqrt{y} = \psi_1(x, y, z) \quad &\Rightarrow \quad \psi_1(1, y, z) = -1 + \sqrt{y} = \bar{\psi}_1 \quad \Rightarrow \\ y = (1 + \bar{\psi}_1)^2 = \lambda_2, \\ -\sqrt{x} + \sqrt{z} = \psi_2(x, y, z) \quad &\Rightarrow \quad \psi_2(1, y, z) = -1 + \sqrt{z} = \bar{\psi}_2 \quad \Rightarrow \\ z = (1 + \bar{\psi}_2)^2 = \lambda_3, \end{aligned}$$

$$u = \varphi(\lambda_2, \lambda_3) = \varphi\left[\left(1+\sqrt{y}-\sqrt{x}\right)^2 - \left(1+\sqrt{z}-\sqrt{x}\right)^2\right].$$

Given that

$$u(1,y,z) = \varphi(y-x) = y-x,$$

we finally obtain

$$u = \left(1+\sqrt{y}-\sqrt{x}\right)^2 - \left(1+\sqrt{z}-\sqrt{x}\right)^2.$$

Problem 250

Find the solution $z = z(x,y)$ of equation

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0,$$

that satisfies the condition

$$z(x,0) = x^2.$$

Solution

This task is the Cauchy problem.

Let us find first the complete solution. The observed equation is a homogeneous linear first order partial equation.We assign to it the system of ordinary equations

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{-x} = \frac{\mathrm{d}z}{0}.$$

From this system we obtain two first integrals

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{-x} \quad \Rightarrow \quad -\int x\,\mathrm{d}x = \int y\,\mathrm{d}y \quad \Rightarrow$$

$$x^2 + y^2 = c_1$$

and, as $\mathrm{d}z/0$, the second first integral

$$z = c_2,$$

where c_1 and c_2 arbitrary constants. As we have already mentioned, if the first integrals are $\psi_1 = c_1$ and $\psi_2 = c_2$, than any differentiable function $F(\psi_1, \psi_2) = 0$ is also a first integral. In this task, this means that the solution (general solution) is

$$z = F\left(x^2 + y^2\right),$$

where $F \in C^1$.

The Cauchy integral (solution of the Cauchy problem) is obtained from the condition

$$z(x,0) = F(x^2) = x^2 \quad \Rightarrow \quad F(x) = x,$$

so we finally obtain the Cauchy integral

$$z(x,y) = x^2 + y^2.$$

Problem 251

Solve the equation

$$\left(1+\sqrt{z-x-y}\right)\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=2.$$

Solution

The corresponding system of ordinary equations in this case is

$$\frac{\mathrm{d}x}{(1+\sqrt{z-x-y})}=\frac{\mathrm{d}y}{1}=\frac{\mathrm{d}z}{2}.$$

From the properties of proportionality, it follows that

$$\frac{-\mathrm{d}x}{-(1+\sqrt{z-x-y})}=\frac{-\mathrm{d}y}{-1}=\frac{\mathrm{d}z}{2}=\frac{\mathrm{d}z-\mathrm{d}x-\mathrm{d}y}{2-1-1-\sqrt{z-x-y}}$$

from where we obtain

$$\frac{-\mathrm{d}x}{-(1+\sqrt{z-x-y})}=\frac{-\mathrm{d}y}{-1}=\frac{\mathrm{d}z}{2}=\frac{\mathrm{d}(z-x-y)}{-\sqrt{z-x-y}}.$$

The first integrals are obtained from

$$\frac{\mathrm{d}y}{1}=\frac{\mathrm{d}z}{2}\quad\Rightarrow\quad 2y=z+c_o$$

i

$$\frac{\mathrm{d}y}{1}=\frac{\mathrm{d}(z-x-y)}{-\sqrt{z-x-y}}\quad\Rightarrow\quad y=-2\sqrt{z-x-y}+c_1,$$

that is

$$y+2\sqrt{z-x-y}=c_1,$$

and thus the general solution is

$$f\left(2y-z,y+2\sqrt{z-x-y}\right)=0.$$

f is an arbitrary differentiable function of corresponding arguments.

R Note[15] that a solution is also the function $z=x+y$. Namely, given that $\frac{\partial z}{\partial x}=1$ and $\frac{\partial z}{\partial y}=1$, it is obvious that this function z also satisfies the initial partial equation. This solution is the so-called **singular solution**.

Problem 252

Determine the integration factor $v=v(x,y)$ so that the expression

$$\left(2x^3y-y^2\right)\mathrm{d}x-\left(2x^4+xy\right)\mathrm{d}y=0$$

[15] Prof. Arpad Takači pointed out to this solution.

becomes a total differential, and then integrate it.

Solution

The condition of integrability (7.46), in this case ($R = 0$), is reduced to the linear partial equation

$$v\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = P\frac{\partial v}{\partial y} - Q\frac{\partial v}{\partial x},$$

to which we assign a system of differential equation

$$\frac{\mathrm{d}x}{-Q} = \frac{\mathrm{d}y}{P} = \frac{\mathrm{d}v}{v(Q_x - P_y)}. \tag{7.296}$$

In our case: $P = 2x^3y - y^2$, $Q = -2x^4 - xy$, from where we obtain $P_y = 2x^3 - 2y$ and $Q_x = -8x^3 - y$, that is

$$Q_x - P_y = -10x^3 + y. \tag{7.297}$$

Now (7.296) can be written in the form

$$\frac{\mathrm{d}v}{v} = \frac{Q_x - P_y}{-Q}\mathrm{d}x = \frac{Q_x - P_y}{P}\mathrm{d}y,$$

that is, when we replace (7.297)

$$\frac{\mathrm{d}v}{v} = \frac{-10x^3 + y}{2x^4 + xy}\mathrm{d}x = \frac{-10x^3 + y}{2x^3y - y^2}\mathrm{d}y. \tag{7.298}$$

Let us first determine the quotients

$$\left(2x^4 + xy\right) : \left(-10x^3 + y\right) = -\frac{1}{5}x + \frac{6/5xy}{-10x^3 + y}, \tag{7.299}$$

$$\left(2x^3y - y^2\right) : \left(-10x^3 + y\right) = -\frac{1}{5}y - \frac{4/5y^2}{-10x^3 + y}. \tag{7.300}$$

Let us first analyze the remainders of divisions in relations (7.299) and (7.300). If we multiply the remainder in (7.299) by $4y$, and the remainder in (7.300) by $6x$ their absolute value become the same. The, bearing in mind that

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}x + \mathrm{d}y}{a + b},$$

we can eliminate the remainders by appropriate adjustments. If we proceed like that , we obtain

$$\frac{\mathrm{d}v}{v} = \frac{4y\,\mathrm{d}x}{-4/5xy + (24/5xy^2)/(-10x^3 + y)} = \frac{6x\,\mathrm{d}y}{-6/5xy - (24/5xy^2)/(-10x^3 + y)} =$$
$$= \frac{4y\,\mathrm{d}x + 6x\,\mathrm{d}y}{-10/5xy} \quad \Rightarrow$$

$$\frac{\mathrm{d}v}{v} = -2\frac{\mathrm{d}x}{x} - 3\frac{\mathrm{d}y}{y} \quad \Rightarrow \quad \ln v = \ln x^{-2} + \ln y^{-3} + \ln c = \ln cx^{-2}y^{-3} \quad \Rightarrow$$

$$v = cx^{-2}y^{-3}.$$

Given that this is true for an arbitrary constant c ($c > 0$), we shall assume that $c = 1$, i.e.

$$v = x^{-2}y^{-3}.$$

Thus, the expression

$$x^{-2}y^{-3}\left(2x^3y - y^2\right)\,\mathrm{d}x - x^{-2}y^{-3}\left(2x^4 + xy\right)\,\mathrm{d}y = 0$$

represents the total differential of a function u, i.e.

$$\mathrm{d}u = x^{-2}y^{-3}\left(2x^3y - y^2\right)\,\mathrm{d}x - x^{-2}y^{-3}\left(2x^4 + xy\right)\,\mathrm{d}y = 0. \tag{7.301}$$

From here we obtain

$$\begin{aligned} u &= \int_{x_0}^{x}\left(2xy^{-2} - x^{-2}y^{-1}\right)\,\mathrm{d}x - \int_{y_0}^{y}\left(2x^2y^{-3} + x^{-1}y^{-2}\right)\,\mathrm{d}y = \\ &= \left(x^2y^{-2} + y^{-1}x^{-1}\right)\Big|_{x_0}^{x} - \left(-x^2y^{-2} - y^{-1}x^{-1}\right)\Big|_{y_0}^{y} = \\ &= 2x^2y^{-2} + 2x^{-}y^{-1} + c. \end{aligned}$$

Check.

$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = \left(4xy^{-2} - 2x^{-2}y^{-1}\right)\mathrm{d}x + \left(-4x^2y^{-3} - 2x^{-1}y^{-2}\right)\mathrm{d}y,$$

and, according to (7.301) this is equal to zero, i.e. $\mathrm{d}u = 0$.

Problem 253

Determine the complete solution and singular solution (if it exists) for equation

$$z = xp + yq + pq.$$

Solution

This is a non-linear partial differential equation. We shall solve it using the Lagrange–Charpit method. We shall us the conditions (7.80). In our case, given that $f = xp + yq + pq - z = 0$, it follows

$$\frac{\partial f}{\partial p} = x + q \quad \frac{\partial f}{\partial q} = y + p \quad \frac{\partial f}{\partial x} = p \quad \frac{\partial f}{\partial y} = q \quad \frac{\partial f}{\partial z} = -1$$

and the equation (7.80) becomes

$$\frac{\mathrm{d}x}{x+q} = \frac{\mathrm{d}y}{y+p} = \frac{\mathrm{d}z}{xp+yq+pq} = \frac{\mathrm{d}p}{-p+p} = \frac{\mathrm{d}q}{-q+q}$$

that is

$$\frac{\mathrm{d}x}{x+q} = \frac{\mathrm{d}y}{y+p} = \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}p}{0} = \frac{\mathrm{d}q}{0}$$

from where we obtain the first two integrals

$$p = const. = a \quad \text{i} \quad q = const. = b.$$

Substituting in the initial equation we obtain the complete solution

$$z = ax + by + ab.$$

Let us look now for the singular solution. If it exists, it is obtained from the system of equations (7.54). In our case

$$g = ax + by + ab - z = 0, \quad \frac{\partial g}{\partial a} = x + b \quad \text{i} \quad \frac{\partial g}{\partial b} = -y + a$$

from where we obtain

$$b = -x \quad \text{i} \quad a = y.$$

Thus, there exists a singular solution

$$xy + z = 0.$$

For the set of examples, which follow without text, determine the type of partial equation and reduce it to the canonical form:

Problem 254

$$\frac{\partial^2 u}{\partial x^2} - 2\sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 u}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0.$$

Solution

The discriminant of the characteristic equation, given that $a_{12} = -\sin x$, $a_{11} = 1$ and $a_{22} = -\cos^2 x$, is

$$D = a_{12}^2 - a_{11}a_{22} = \sin^2 x + \cos^2 x = 1 > 0,$$

thus positive in the entire plane x, y. The observed equation is hence of the **hyperbolic type** in the entire x, y plane.

It has two real characteristics, which are obtained by solving the characteristic equation

$$a_{11}\mathrm{d}y^2 - 2a_{12}\mathrm{d}x\mathrm{d}y + a_{22}\mathrm{d}x^2 = 0,$$

which in this case has the form

$$\mathrm{d}y^2 + 2\sin x\mathrm{d}x\mathrm{d}y - \cos^2 x\mathrm{d}x^2 = 0,$$

from where we obtain

$$(y')^2 + 2\sin x y' - cos^2 x = 0 \quad \Rightarrow$$

$$y'_{1,2} = \frac{-2\sin x \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2} = -\sin x \pm 1.$$

The characteristics are

$$y_1' = -\sin x + 1 \quad \Rightarrow \quad y = x + \cos x + c_1 \quad \text{that is}$$
$$y_2' = -\sin x + 1 \quad \Rightarrow \quad y = -x + \cos x + c_2.$$

the substitutions are

$$c_1 = y - x - \cos x = \xi \quad \text{i}$$
$$c_2 = y + x - \cos x = \eta.$$

The corresponding partial derivatives, necessary for calculating the new coefficients, are

$$\frac{\partial \xi}{\partial x} = -1 + \sin x, \quad \frac{\partial \xi}{\partial y} = 1, \quad \frac{\partial \eta}{\partial x} = 1 + \sin x, \quad \frac{\partial \eta}{\partial y} = 1,$$
$$\frac{\partial^2 \xi}{\partial x^2} = \cos x, \quad \frac{\partial^2 \xi}{\partial x \partial y} = 0, \quad \frac{\partial^2 \xi}{\partial y^2} = 0,$$
$$\frac{\partial^2 \eta}{\partial x^2} = \cos x, \quad \frac{\partial^2 \eta}{\partial x \partial y} = 0, \quad \frac{\partial^2 \eta}{\partial y^2} = 0.$$

Check of the Jacobian:

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 + \sin x & 1 \\ 1 + \sin x & 1 \end{vmatrix} = -2 \neq 0.$$

Calculation of new coefficients:

$$\bar{a}_{11} = a_{11} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2a_{12} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + a_{22} \left(\frac{\partial \xi}{\partial y} \right)^2 =$$
$$= (-1 + \sin x)^2 + 2(-\sin x)(-1 + \sin x) + (-\cos^2 x) = 0,$$

$$\bar{a}_{22} = a_{11} \left(\frac{\partial \eta}{\partial x} \right)^2 + 2a_{12} \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + a_{22} \left(\frac{\partial \eta}{\partial y} \right)^2 =$$
$$= (1 + \sin x)^2 + 2(-\sin x)(1 + \sin x) + (-\cos^2 x) = 0,$$

$$\bar{a}_{12} = a_{11} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + a_{12} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + a_{21} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} + a_{22} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} =$$
$$= (-1 + \sin x)(1 + \sin x) + (-\sin x)\left[(-1 + \sin x) + (1 + \sin x)\right] + (-\cos^2 x) = -2.$$

Given that $a_1 = 0$ and $a_2 = -\cos x$, we obtain:

$$\bar{a}_1 = a_1 \frac{\partial \xi}{\partial x} + a_2 \frac{\partial \xi}{\partial y} + a_{11} \frac{\partial^2 \xi}{\partial x^2} + 2a_{12} \frac{\partial^2 \xi}{\partial x \partial y} + a_{22} \frac{\partial^2 \xi}{\partial y^2} =$$
$$= (-\cos x) + \cos x =$$
$$= 0,$$

$$\bar{a}_2 = a_1\frac{\partial\eta}{\partial x} + a_2\frac{\partial\eta}{\partial y} + a_{11}\frac{\partial^2\eta}{\partial x^2} + 2a_{12}\frac{\partial^2\eta}{\partial x\partial y} + a_{22}\frac{\partial^2\eta}{\partial y^2} =$$
$$= (-\cos x) + \cos x =$$
$$= 0.$$

Given that $b = 0$ it follows that also $\bar{b} = 0$, and the transformed equation, when $u(x,y) \to v(\xi,\eta)$, finally obtains the form

$$\bar{a}_{12}\frac{\partial^2 v}{\partial\xi\partial\eta} = 0 \quad \Rightarrow \quad \frac{\partial^2 v}{\partial\xi\partial\eta} = 0.$$

Problem 255

$$\frac{\partial^2 u}{\partial x^2} - 2\cos x\frac{\partial^2 u}{\partial x\partial y} - (3+\sin^2 x)\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y} = 0.$$

Solution

The old coefficients

$$a_{11} = 1, \quad a_{12} = -\cos x, \quad a_{22} = -(3+\sin^2 x), \quad a_1 = 0, \quad a_2 = -y.$$

The discriminant
$$D = \cos^2 x + 3 + \sin^2 x = 4 > 0.$$

Given that the discriminant is greater than zero in the entire plane x,y, this equation is of **hyperbolic type** in the entire plane x,y.

Given that $D > 0$ we have two real characteristics.

The characteristic equation, in this case, has the form

$$\mathrm{d}y^2 + 2\cos x\,\mathrm{d}x\mathrm{d}y - (3+\sin^2 x)\mathrm{d}x^2 = 0 \quad \Rightarrow$$

$$y'_{1,2} = \frac{-2\cos x \pm \sqrt{4\cos^2 x + 4(3+\sin^2 x)}}{2} = -\cos x \pm 2.$$

The characteristics are

$$y = -\sin x + 2x + c_1,$$
$$y = -\sin x - 2x + c_2.$$

The substitutions are

$$c_1 = \xi = y - 2x + \sin x,$$
$$c_2 = \eta = y + 2x + \sin x.$$

The corresponding partial derivatives are

$$\frac{\partial \xi}{\partial x} = -2+\cos x, \quad \frac{\partial \xi}{\partial y} = 1, \quad \frac{\partial \eta}{\partial x} = 2+\cos x, \quad \frac{\partial \eta}{\partial y} = 1,$$
$$\frac{\partial^2 \xi}{\partial x^2} = -\sin x, \quad \frac{\partial^2 \xi}{\partial y^2} = 0, \quad \frac{\partial^2 \xi}{\partial x \partial y} = 0,$$
$$\frac{\partial^2 \eta}{\partial x^2} = -\sin x, \quad \frac{\partial^2 \xi}{\partial y^2} = 0, \quad \frac{\partial^2 \xi}{\partial x \partial y} = 0.$$

Check of the Jacobian:

$$J = \begin{vmatrix} -2+\cos x & 1 \\ 2+\cos x & 1 \end{vmatrix} = -4 \neq 0.$$

Calculation of new coefficients:

$$\bar{a}_{11} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2 =$$
$$= (-2+\cos x)^2 + 2(-\cos x)(-2+\cos x) + (-3-\sin^2 x) = 1-(\cos^2 x+\sin^2 x) =$$
$$= 0,$$

$$\bar{a}_{22} = a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2 =$$
$$= (2+\cos x)^2 + 2(-\cos x)(2+\cos x) + (-3-\sin^2 x) = 1-(\cos^2 x+\sin^2 x) =$$
$$= 0,$$

$$\bar{a}_{12} = a_{11}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + a_{12}\left(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + a_{22}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} =$$
$$= (-2+\cos x)(2+\cos x) + (-\cos x)(-2+\cos x+2+\cos x) + (-3-\sin^2 x) =$$
$$= -8.$$

$$\bar{a}_1 = a_1\frac{\partial \xi}{\partial x} + a_2\frac{\partial \xi}{\partial y} + a_{11}\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 \xi}{\partial x \partial y} + a_{22}\frac{\partial^2 \xi}{\partial y^2} =$$
$$= -y-\sin x.$$

From transformations

$$\xi = y+\sin x - 2x,$$
$$\eta = y+\sin x + 2x,$$

it follows that $y+\sin x = 1/2(\xi+\eta)$, and thus

$$\bar{a}_1 = \frac{1}{2}(\xi+\eta).$$

$$\bar{a}_2 = a_1\frac{\partial \eta}{\partial x} + a_2\frac{\partial \eta}{\partial y} + a_{11}\frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial^2 \eta}{\partial x \partial y} + a_{22}\frac{\partial^2 \eta}{\partial y^2} = -y-\sin x =$$
$$= \frac{1}{2}(\xi+\eta).$$

The transformed partial equation $u(x,y) \to v(\xi,\eta)$ is now of the form

$$\bar{a}_{11}\frac{\partial^2 v}{\partial \xi^2} + 2\bar{a}_{12}\frac{\partial^2 v}{\partial \xi \partial \eta} + \bar{a}_{22}\frac{\partial^2 v}{\partial \eta^2} + \bar{a}_1\frac{\partial v}{\partial \xi} + \bar{a}_2\frac{\partial v}{\partial \eta} + \bar{b}v + \bar{c} =$$
$$-2\cdot 8\frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{1}{2}(\xi+\eta)\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = 0 \quad \Rightarrow$$

$$\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{1}{32}(\xi+\eta)\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = 0.$$

Problem 256

$$\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 10\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0.$$

Solution

$$a_{11} = 1, \quad a_{12} = 3, \quad a_{22} = 10, \quad a_1 = 1, \quad a_2 = 3$$
$$D = a_{12}^2 - a_{11}a_{22} = 9 - 10 = -1 < 0.$$

Given that $D < 0$ in the entire plane xy, the equation is of **elliptic type**.

Characteristic equation is

$$a_{11}(y')^2 - 2a_{12}(y') + a_{22} = 0 \quad \Rightarrow \quad (y')_{1,2} = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

We have two solution – complex conjugated.

The characteristics are

$$\frac{\mathrm{dy}}{\mathrm{dx}} = 3 \pm and \quad \Rightarrow \quad \begin{cases} \mathrm{dy} = (3+i)\mathrm{dx} & \Rightarrow \quad y = (3+i)x + c_1 \\ \mathrm{dy} = (3-i)\mathrm{dx} & \Rightarrow \quad y = (3-i)x + c_2. \end{cases}$$

Transformations (real and imaginary parts of the characteristics) are

$$\xi = y - 3x,$$
$$\eta = -x.$$

Partial derivatives are

$$\frac{\partial \xi}{\partial x} = -3, \quad \frac{\partial \xi}{\partial y} = 1, \quad \frac{\partial \eta}{\partial x} = -1, \quad \frac{\partial \eta}{\partial y} = 0,$$
$$\frac{\partial^2 \xi}{\partial x^2} = 0, \quad \frac{\partial^2 \xi}{\partial y^2} = 0, \quad \frac{\partial^2 \xi}{\partial x \partial y} = 0,$$
$$\frac{\partial^2 \eta}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial y^2} = 0, \quad \frac{\partial^2 \eta}{\partial x \partial y} = 0.$$

The Jacobian is

$$J = \begin{vmatrix} -3 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0.$$

Calculation of new coefficients

$$\begin{aligned}\bar{a}_{11} = a_{11}\left(\frac{\partial\xi}{\partial x}\right)^2 + 2a_{12}\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\xi}{\partial y}\right) + a_{22}\left(\frac{\partial\xi}{\partial y}\right)^2 = \\ = (-3)^2 + 2(3)(-3) + 10 = \\ = 1,\end{aligned}$$

$$\begin{aligned}\bar{a}_{22} = a_{11}\left(\frac{\partial\eta}{\partial x}\right)^2 + 2a_{12}\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\eta}{\partial y}\right) + a_{22}\left(\frac{\partial\eta}{\partial y}\right)^2 = \\ = (-1)^2 + 2(3)(-1)0 + 0 = \\ = 1,\end{aligned}$$

$$\begin{aligned}\bar{a}_{12} = a_{11}\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right) + a_{12}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + a_{22}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = \\ = (-3)(-1) + 3[(-3)0 + 1(-1)] + 10\cdot 0 = 3 - 3 \\ = 0,\end{aligned}$$

$$\begin{aligned}\bar{a}_1 = a_1\frac{\partial\xi}{\partial x} + a_2\frac{\partial\xi}{\partial y} + a_{11}\frac{\partial^2\xi}{\partial x^2} + 2a_{12}\frac{\partial^2\xi}{\partial x\partial y} + a_{22}\frac{\partial^2\xi}{\partial y^2} = \\ = 1(-3) + 3\cdot 1 = 0,\end{aligned}$$

$$\begin{aligned}\bar{a}_2 = a_1\frac{\partial\eta}{\partial x} + a_2\frac{\partial\eta}{\partial y} + a_{11}\frac{\partial^2\eta}{\partial x^2} + 2a_{12}\frac{\partial^2\eta}{\partial x\partial y} + a_{22}\frac{\partial^2\eta}{\partial y^2} = \\ = 1(-1) + 3\cdot 0 = -1.\end{aligned}$$

Transformed equation $u(x,y) \to v(\xi,\eta)$

$$\bar{a}_{11}\frac{\partial^2 v}{\partial\xi^2} + 2\bar{a}_{12}\frac{\partial^2 v}{\partial\xi\partial\eta} + \bar{a}_{22}\frac{\partial^2 v}{\partial\eta^2} + \bar{a}_1\frac{\partial v}{\partial\xi} + \bar{a}_2\frac{\partial v}{\partial\eta} + \bar{b}v + \bar{c} = 0 \quad \Rightarrow$$

gains the form

$$\frac{\partial v}{\partial\xi} + \frac{\partial^2 v}{\partial\eta^2} - \frac{\partial v}{\partial\eta} = 0.$$

Problem 257

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} = 0, \quad x \neq 0.$$

Solution

$$a_{11} = 1, \quad a_{12} = 0, \quad a_{22} = x^2, \quad a_1 = 0, \quad a_2 = 0$$
$$D = a_{12}^2 - a_{11}a_{22} = -x^2 < 0.$$

Given that $D < 0$ in the entire plane xy, except at point $x = 0$, which is excluded, the equation is of **elliptic type**.

The characteristic equation is

$$a_{11}(y')^2 - 2a_{12}(y') + a_{22} = 0 \quad \Rightarrow \quad (y')_{1,2} = \pm xi.$$

We have two solutions – complex conjugated.

The characteristics are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm xi \quad \Rightarrow \quad \begin{cases} \mathrm{d}y = xi\mathrm{d}x & \Rightarrow \quad y = \frac{1}{2}xi + c_1, \\ \mathrm{d}y = -xi\mathrm{d}x & \Rightarrow \quad y = -\frac{1}{2}xi + c_2. \end{cases}$$

Transformations (real and imaginary part of the characteristics)

$$\xi = y,$$
$$\eta = \frac{1}{2}x.$$

The partial derivatives are

$$\frac{\partial \xi}{\partial x} = 0, \quad \frac{\partial \xi}{\partial y} = 1, \quad \frac{\partial \eta}{\partial x} = x, \quad \frac{\partial \eta}{\partial y} = 0,$$
$$\frac{\partial^2 \xi}{\partial x^2} = 0, \quad \frac{\partial^2 \xi}{\partial y^2} = 0, \quad \frac{\partial^2 \xi}{\partial x \partial y} = 0,$$
$$\frac{\partial^2 \eta}{\partial x^2} = 1, \quad \frac{\partial^2 \eta}{\partial y^2} = 0, \quad \frac{\partial^2 \eta}{\partial x \partial y} = 0.$$

The Jacobian is

$$J = \begin{vmatrix} -x & 1 \\ x & 0 \end{vmatrix} = -x \neq 0,$$

because $x = 0$ is excludeed.

Calculation of new coefficients:

$$\bar{a}_{11} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2 =$$
$$= (0)^2 + 2(0)(-x) + x^2 = x^2,$$

$$\bar{a}_{22} = a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2 =$$
$$= (x)^2 + 2(0)(x)0 + x^2(0)^2 = x^2,$$

$$\bar{a}_{12} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + a_{22}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0,$$

$$\bar{a}_1 = a_1\frac{\partial \xi}{\partial x} + a_2\frac{\partial \xi}{\partial y} + a_{11}\frac{\partial^2 \xi}{\partial x^2} + 2a_{12}\frac{\partial^2 \xi}{\partial x \partial y} + a_{22}\frac{\partial^2 \xi}{\partial y^2} = 0,$$

$$\bar{a}_2 = a_1\frac{\partial \eta}{\partial x} + a_2\frac{\partial \eta}{\partial y} + a_{11}\frac{\partial^2 \eta}{\partial x^2} + 2a_{12}\frac{\partial^2 \eta}{\partial x \partial y} + a_{22}\frac{\partial^2 \eta}{\partial y^2} =$$
$$= 1(1) = 1.$$

The transformed equation $u(x,y) \to v(\xi,\eta)$

$$\bar{a}_{11}\frac{\partial^2 v}{\partial \xi^2} + 2\bar{a}_{12}\frac{\partial^2 v}{\partial \xi \partial \eta} + \bar{a}_{22}\frac{\partial^2 v}{\partial \eta^2} + \bar{a}_1\frac{\partial v}{\partial \xi} + \bar{a}_2\frac{\partial v}{\partial \eta} + \bar{b}v + \bar{c} = 0 \quad \Rightarrow$$

gains the form

$$\frac{\partial v}{\partial \xi} + \frac{\partial^2 v}{\partial \eta^2} + \frac{1}{2\eta}\frac{\partial v}{\partial \eta} = 0,$$

where we have used that $x^2 = 2\eta$.

Problem 258

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2u = 0.$$

Solution

$$a_{11} = 1, \quad a_{12} = -1, \quad a_{22} = 1, \quad a_1 = 3, \quad a_2 = 1,$$
$$D = a_{12}^2 - a_{11}a_{22} = 1 - 1 = 0.$$

Given that $D = 0$ in the entire plane xy, the equation is of **parabolic type**.

The characteristic equation is

$$a_{11}(y')^2 - 2a_{12}(y') + a_{22} = 0 \quad \Rightarrow \quad (y')^2 + 2(y') + 1 = 0 \quad \Rightarrow \quad (y')_{1,2} = -1.$$

It has only one solution – one characteristic.

The characteristic is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 \quad \Rightarrow \quad \mathrm{d}y = -\mathrm{d}x \quad \Rightarrow \quad y = -x + c_1.$$

The transformations are

$$\xi = x + y,$$
$$\eta = \eta(x,y) = y.$$

Given that η is an arbitrary function, we chose it to be as simple as possible for further use, but also that the Jacobian is not equal to zero.

The partial derivatives are

$$\frac{\partial \xi}{\partial x}=1, \quad \frac{\partial \xi}{\partial y}=1, \quad \frac{\partial \eta}{\partial x}=0, \quad \frac{\partial \eta}{\partial y}=1,$$
$$\frac{\partial^2 \xi}{\partial x^2}=0, \quad \frac{\partial^2 \xi}{\partial y^2}=0, \quad \frac{\partial^2 \xi}{\partial x \partial y}=0,$$
$$\frac{\partial^2 \eta}{\partial x^2}=0, \quad \frac{\partial^2 \eta}{\partial y^2}=0, \quad \frac{\partial^2 \eta}{\partial x \partial y}=0.$$

The Jacobian is

$$J=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=1\neq 0.$$

Calculation new coefficients:

$$\begin{aligned}\bar{a}_{11} &= a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2+2a_{12}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right)+a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2= \\ &= 1(1)^2+2(-1)(1)+1^2=0,\end{aligned}$$

$$\begin{aligned}\bar{a}_{22} &= a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2+2a_{12}\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)+a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2= \\ &= (0)^2-2(0)(1)+1^2=1,\end{aligned}$$

$$\begin{aligned}\bar{a}_{12} &= a_{11}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)+a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+a_{22}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}= \\ &= 1(0)+(-1)(1)+1=0,\end{aligned}$$

$$\begin{aligned}\bar{a}_{1} &= a_1\frac{\partial \xi}{\partial x}+a_2\frac{\partial \xi}{\partial y}+a_{11}\frac{\partial^2 \xi}{\partial x^2}+2a_{12}\frac{\partial^2 \xi}{\partial x\partial y}+a_{22}\frac{\partial^2 \xi}{\partial y^2}= \\ &= 3\cdot 1+1\cdot 1=4,\end{aligned}$$

$$\begin{aligned}\bar{a}_{2} &= a_1\frac{\partial \eta}{\partial x}+a_2\frac{\partial \eta}{\partial y}+a_{11}\frac{\partial^2 \eta}{\partial x^2}+2a_{12}\frac{\partial^2 \eta}{\partial x\partial y}+a_{22}\frac{\partial^2 \eta}{\partial y^2}= \\ &= 0+1+0+0+0=1.\end{aligned}$$

The transformed equation $u(x,y)\to v(\xi,\eta)$

$$\bar{a}_{11}\frac{\partial^2 v}{\partial \xi^2}+2\bar{a}_{12}\frac{\partial^2 v}{\partial \xi\partial \eta}+\bar{a}_{22}\frac{\partial^2 v}{\partial \eta^2}+\bar{a}_1\frac{\partial v}{\partial \xi}+\bar{a}_2\frac{\partial v}{\partial \eta}+\bar{b}v+\bar{c}=0 \quad \Rightarrow$$

gains the form

$$\frac{\partial^2 v}{\partial \eta^2}+4\frac{\partial v}{\partial \xi}+\frac{\partial v}{\partial \eta}+2v=0.$$

Problem 259

$$\sin^2 x \frac{\partial^2 u}{\partial x^2} - 2y \sin x \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution

$$a_{11} = \sin^2 x, \quad a_{12} = -y \sin x, \quad a_{22} = y^2, \quad a_1 = 0, \quad a_2 = 0$$

$$D = a_{12}^2 - a_{11}a_{22} = y^2 \sin^2 x - (\sin^2 x)y^2 = 0.$$

Given that $D = 0$ in the entire plane xy, the equation is of **parabolic type**.

The characteristic equation

$$a_{11}(y')^2 - 2a_{12}(y') + a_{22} = 0 \quad \Rightarrow \quad (y')^2 - 2(-y\sin x)(y') + y^2 = 0 \quad \Rightarrow$$
$$(y')_{1,2} = -\frac{y}{\sin x}.$$

It has only one solution – one characteristic.

The characteristic is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{\sin x} \quad \Rightarrow \quad \frac{\mathrm{d}y}{y} = -\frac{\mathrm{d}x}{\sin x} \quad \Rightarrow \quad y \mathrm{tg}\frac{x}{2} = c_1.$$

We have used here

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\mathrm{d}x = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\mathrm{d}x =$$
$$= \ln \mathrm{tg}\frac{x}{2}.$$

The transformations are

$$\xi = y\mathrm{tg}\frac{x}{2},$$
$$\eta = \eta(x,y) = y.$$

R Note that , given that η is an arbitrary function, we chose it to be as simple as possible for further use, but also that the Jacobian is not equal to zero. The simplest choice would be $\eta = const.$, but then $J = 0$. The next choices would be $\eta = x$ or $\eta = y$. We have chosen the latter.

The partial derivatives are

$$\frac{\partial \xi}{\partial x} = \frac{1}{2}y\cos^{-2}\frac{x}{2}, \quad \frac{\partial \xi}{\partial y} = \mathrm{tg}\frac{x}{2}, \quad \frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial y} = 1,$$
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{2}y\cos^{-3}\frac{x}{2}\sin\frac{x}{2}, \quad \frac{\partial^2 \xi}{\partial y^2} = 0, \quad \frac{\partial^2 \xi}{\partial x \partial y} = \frac{1}{2}\cos^{-2}\frac{x}{2},$$
$$\frac{\partial^2 \eta}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial y^2} = 0, \quad \frac{\partial^2 \eta}{\partial x \partial y} = 0.$$

The Jacobian is

$$J = \begin{vmatrix} \frac{1}{2} y \cos^{-2} \frac{x}{2} & \operatorname{tg} \frac{x}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2} y \cos^{-2} \frac{x}{2} \neq 0.$$

Calculation of new coefficients

$$\bar{a}_{11} = a_{11} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2a_{12} \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) + a_{22} \left(\frac{\partial \xi}{\partial y} \right)^2 =$$
$$= \sin^2 x \left(\frac{1}{2} y \cos^{-2} \frac{x}{2} \right)^2 + 2(-y \sin x) \frac{1}{2} y \cos^{-2} \frac{x}{2} \operatorname{tg} \frac{x}{2} + y^2 \operatorname{tg}^2 \frac{x}{2} = 0,$$

$$\bar{a}_{22} = a_{11} \left(\frac{\partial \eta}{\partial x} \right)^2 + 2a_{12} \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + a_{22} \left(\frac{\partial \eta}{\partial y} \right)^2 =$$
$$= 0 + 0 + y^2 =$$
$$= y^2 = \eta^2,$$

$$\bar{a}_{12} = a_{11} \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) + a_{12} \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + a_{22} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} =$$
$$= -y \sin x \frac{1}{2} y \cos^{-2} \frac{1}{2} + y^2 \operatorname{tg} \frac{1}{2} = 0,$$

$$\bar{a}_1 = a_1 \frac{\partial \xi}{\partial x} + a_2 \frac{\partial \xi}{\partial y} + a_{11} \frac{\partial^2 \xi}{\partial x^2} + 2a_{12} \frac{\partial^2 \xi}{\partial x \partial y} + a_{22} \frac{\partial^2 \xi}{\partial y^2} =$$
$$= \sin^2 x \left(\frac{1}{2} y \cos^{-3} \frac{x}{2} \sin \frac{x}{2} \right) + 2(-y \sin x) \frac{1}{2} \cos^{-2} \frac{x}{2} =$$
$$= -2y \operatorname{tg} \frac{x}{2} \cos^2 \frac{x}{2} = -2y \operatorname{tg} \frac{x}{2} \frac{1}{1 + \operatorname{tg}^2 \frac{x}{2}}.$$

Given that (from the transformations)

$$\operatorname{tg} \frac{x}{2} = \frac{\xi}{\eta},$$

for $\bar{a}_1$ we obtain:

$$\bar{a}_1 = -2\xi \frac{\eta^2}{\xi^2 + \eta^2}. \tag{7.302}$$

$$\bar{a}_2 = a_1 \frac{\partial \eta}{\partial x} + a_2 \frac{\partial \eta}{\partial y} + a_{11} \frac{\partial^2 \eta}{\partial x^2} + 2a_{12} \frac{\partial^2 \eta}{\partial x \partial y} + a_{22} \frac{\partial^2 \eta}{\partial y^2} =$$
$$= 0 + 0 + 0 + 0 + 0 = 0.$$

The transformed equation $u(x, y) \to v(\xi, \eta)$

$$\bar{a}_{11} \frac{\partial^2 v}{\partial \xi^2} + 2\bar{a}_{12} \frac{\partial^2 v}{\partial \xi \partial \eta} + \bar{a}_{22} \frac{\partial^2 v}{\partial \eta^2} + \bar{a}_1 \frac{\partial v}{\partial \xi} + \bar{a}_2 \frac{\partial v}{\partial \eta} + \bar{b} v + \bar{c} = 0 \quad \Rightarrow$$

gains the form

$$\frac{\partial^2 v}{\partial \eta^2} - \frac{2\xi}{\xi^2 + \eta^2} \frac{\partial v}{\partial \xi} = 0.$$

Problem 260

As a characteristic example of a partial differential equation of second order, which comes down to Bessel equation, let us observe the equation

$$\Delta u + k^2 u = 0, \tag{7.303}$$

on or inside of a circle (two variables) or, on or inside a cylinder (three variables). Let us find its solution.

Solution

Using the polar coordinates (two variables: r, φ) the equation (7.303) can be written in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0. \tag{7.304}$$

Assume, further, that the solution u can be represented in the form

$$u = R(r)\cdot\Phi(\varphi), \tag{7.305}$$

and the equation (7.304) is decomposed to two ordinary differential equations

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(-\frac{\lambda}{r^2} + k^2\right)R = 0, \tag{7.306}$$

$$\Phi'' + \lambda\Phi = 0. \tag{7.307}$$

As we have shown previously, the equation (7.307) yields the dependence $\lambda = n^2$. Let us now introduce the substitution $x = kr$, and equation (7.306) becomes the Bessel equation:

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xy') + \left(1 - \frac{n^2}{x^2}\right)y = 0, \tag{7.308}$$

$$R(r) = y(x) = y(kr). \tag{7.309}$$

In case of radial symmetry ($n = 0$) it comes down to the Bessel equation of zero order

$$y'' + \frac{1}{x}y' + y = 0. \tag{7.310}$$

Problem 261

Find the solution of equation

$$u_{tt} = a^2 u_{xx},$$

that satisfies the boundary

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le x \le 1$$

and initial conditions

$$u(x,0) = x, \quad u_t(x,0) = 0.$$

Solution

The solution of the equation is of the form

$$u = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\pi n}{l} at\right) + B_n \sin\left(\frac{\pi n}{l} at\right)\right] \sin\frac{\pi n}{l} x.$$

where

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\cdot\sin\frac{n\pi x}{l}\cdot dx = \frac{2}{l}\int_0^l x\cdot\sin\frac{n\pi x}{l} dx = \frac{2}{l}\int_0^{n\pi} \frac{l}{n\pi}\cdot t\sin t\cdot\frac{l}{n\pi} dt = \frac{2l}{n^2\pi^2}\int_0^{n\pi} t\sin t dt =$$

$$= \frac{2l}{n^2\pi^2}(\sin t - t\cdot\cos t)\Big|_0^{n\pi} = \frac{2l}{n^2\pi^2}(\sin n\pi - n\pi\cdot\cos n\pi - \sin 0 + 0\cdot\cos 0) =$$

$$= \frac{2l}{n^2\pi^2}\cdot n\pi\cdot(-1)^{n+1} = \frac{2l}{n\pi}\cdot(-1)^{n+1}.$$

The required coefficients are of the form:

$$A_n = \frac{2l}{n\pi}\cdot(-1)^{n+1}, \quad B_n = \frac{2}{\pi an}\int_0^l \psi(x)\cdot\sin\frac{n\pi x}{l} dx = \frac{2}{\pi an}\int_0^l 0\cdot\sin\frac{n\pi x}{l} dx = 0.$$

The solution of the given partial differential equation is of the form

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos\frac{an\pi t}{l}\cdot\sin\frac{n\pi x}{l}.$$

Problem 262

Find the solution of equation

$$u_{tt} = a^2 u_{xx},$$

that satisfies boundary

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le x \le l$$

and initial conditions

$$u(x,0) = \pi - x, \quad u_t(x,0) = 0.$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} \left[1 + (-1)^{n+1}\left(1 - \frac{l}{\pi}\right)\right] \cos\frac{an\pi t}{l} \cdot \sin\frac{n\pi x}{l}.$$

Problem 263

Find the solution of equation

$$u_{tt} = a^2 u_{xx},$$

that satisfies boundary

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le x \le l$$

and initial conditions

$$u(x,0) = x + a, \quad u_t(x,0) = 0, \quad \text{where } a = \text{const}.$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[(-1)^{n+1}(l+a) + a\right] \cos\frac{an\pi t}{l} \cdot \sin\frac{n\pi x}{l}.$$

Problem 264

Find the solution of equation

$$u_{tt} = a^2 u_{xx},$$

that satisfies boundary

$$u(0,t) = 0, \quad u(\ell,t) = 0, \quad 0 \le x \le \ell$$

and initial conditions

$$u(x,0) = \begin{cases} \dfrac{h}{x_0}x, & 0 \le x \le x_0, \\[2ex] \dfrac{h(\ell - x)}{\ell - x_0}, & x_0 \le x \le \ell \end{cases}$$

$$u_t(\ell,t) = 0, \quad 0 \le x \le \ell.$$

Solution

$$u(x,t) = \frac{2h\ell^2}{\pi^2 x_0(\ell - x_0)} \sum_{n=1}^{\infty} \sin\frac{n\pi x_0}{\ell} \sin\frac{n\pi x}{\ell} \cos\frac{n\pi a t}{\ell}.$$

Problem 265

Find the solution of equation

$$a^2 u_{xx} = u_t, \quad 0 < x < l, \quad 0 \le t, \tag{7.311}$$

that satisfies initial

$$u(x,0) = \varphi(x) = \begin{cases} x, & 0 \le x \le \frac{l}{2}; \\ l - x, & \frac{l}{2} \le x \le l \end{cases} \tag{7.312}$$

and boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0 \quad 0 \le t. \tag{7.313}$$

Solution

$$C_n = \frac{2}{l}\left(\int_0^{\frac{l}{2}} x \sin\frac{n\pi x}{l} \mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x) \sin\frac{n\pi x}{l} \mathrm{d}x\right). \tag{7.314}$$

For even values of n the constant C_n is equal to zero, and for odd values

$$B_n = \begin{cases} \frac{4l}{n^2\pi^2}, & \text{za } n = 1,5,9,\ldots \\ -\frac{4l}{n^2\pi^2}, & \text{za } n = 3,7,11,\ldots \end{cases} \tag{7.315}$$

and the final solution is

$$u(x,t) = \frac{4l}{\pi^2}\left[\sin\frac{\pi x}{l} e^{-(\frac{a\pi}{l})^2 t} - \frac{1}{9}\sin\frac{3\pi x}{l} e^{-(\frac{3a\pi}{l})^2 t} + \ldots\right]. \tag{7.316}$$

Problem 266

Determine the type of PDE

$$4u_t = u_{xx}, \quad 0 \le x \le 2,\ t > 0, \tag{7.317}$$

and find its solution for the following

- boundary conditions

$$u(0,t)=0,$$
$$u(2,t)=0,$$

- initial condition

$$u(x,0)=2\sin\left(\frac{\pi x}{2}\right)-\sin(\pi x)+4\sin(2\pi x)=f(x).$$

Solution

According to 4° (see p. 362) the equation is of parabolic type.

We will look for its solution in the form (method of separation of variables)

$$u(x;t)=X(x)T(t),$$

and thus teh equation (7.317) becomes

$$4X(x)\dot{T}(t)=X''(x)T(t).$$

R Note that $X\not\equiv 0$ and $T\not\equiv 0$, as we are looking for a non-trivial solution.

If we now divide this equation by XT, we obtain

$$4\frac{\dot{T}}{T}=\frac{X''}{X}=\lambda, \tag{7.318}$$

where λ is a constant.

Given that the left hand side is a function of the independent variable t, and the right hand side of the independent variable x, the equal sign is possible only in the case when they are equal to a constant (λ). Thus, this PDE is decomposed to two ordinary differential equations

$$4\dot{T}-\lambda T=0, \quad \wedge \quad X''-\lambda X=0.$$

Boundary conditions now come down to

$$u(0,t)=X(0)\cdot T(t)=0 \quad \Rightarrow \quad X(0)=0,$$
$$u(2,t)=X(2)\cdot T(t)=0 \quad \Rightarrow \quad X(2)=0.$$

Thus, the first part of the problem is to find values of the constant λ for which we obtain non-trivial solution of the problem:

$$X''+\lambda X=0, \quad X(0)=X(2)=0, \tag{7.319}$$

as well as the corresponding function $X(x)$.

Assume that the solution of the equation (7.319) is of the form

$$X(x) = Ce^{\alpha x} \quad \Rightarrow \quad \alpha^2 Ce^{\alpha x} + \lambda Ce^{\alpha x} = 0 \quad \Rightarrow \quad Ce^{\alpha x}\left(\alpha^2 + \lambda\right) = 0$$
$$\Rightarrow \quad \alpha = \pm\sqrt{-\lambda} \quad \Rightarrow \quad X(x) = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

The analysis (discussion with respect to value λ, namely, is it: positive, 0 or negative) (see discussion on p. 366) shows that the only interesting case is the one where

$$X(x) = C_1 e^{\mu xi} + C_2 e^{-\mu xi} = C_1(\cos\mu x + i\sin\mu x) + C_2(\cos\mu x - i\sin\mu x),$$
$$X(x) = \underbrace{(C_1 + C_2)}A\cos\mu x + \underbrace{(C_1 - C_2)i}B\sin\mu x =$$
$$X(x) = A\cos\mu x + B\sin\mu x.$$

From boundary conditions it follows that

$$X(0) = A \quad \wedge \quad X(2) = B\sin 2\mu = 0.$$

From the last condition we conclude that $B \neq 0$, because otherwise we would obtain the trivial solution ($A = 0, B = 0 \quad \Rightarrow \quad u(x,t) = 0$ for each x and t), and thus, because $B \neq 0$, the following must be true

$$\sin 2\mu = 0 \quad \Rightarrow \quad 2\mu = n\pi \quad \Rightarrow \quad \mu = \frac{n\pi}{2}, \quad n = 1, 2, \ldots,$$
$$\lambda = -\mu^2 = -\left(\frac{n\pi}{2}\right)^2$$

Finally, for the first part of the solution, we obtain

$$X_n(x) = B\sin\frac{n\pi}{2}x.$$

Now the second equation (when λ is determined) becomes

$$4\dot{T} = \lambda T \quad \Rightarrow \quad \frac{\mathrm{d}T}{T} = \frac{\lambda_n}{4} = -\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 \quad \Rightarrow$$
$$\ln T = -\frac{n^2\pi^2}{16}t \quad \Rightarrow \quad T = e^{-\frac{n^2\pi^2}{16}t},$$

and thus

$$u_n = X_n(x)\cdot T_n(t) = B_n \sin\frac{n\pi}{2}x\cdot\exp\left(-\frac{n^2\pi^2}{16}t\right)$$

that is (property 1. - superposition, on p. 356)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)\exp\left(-\frac{n^2\pi^2}{16}t\right)$$

where b_n are constants.

The constants b_n can be determined from the initial conditions, i.e.

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right). \tag{7.320}$$

Thus, it is necessary to expand the function $\varphi(x)$ into a Fourier series, and then, comparing with (7.320), obtain the constants b_n, i.e.

$$
\begin{aligned}
b_n &= \int_0^2 \varphi(x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = \\
&= \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = \\
&= 2\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x - \int_0^2 \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x + \\
&+ 4\int_0^2 \sin(2\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x.
\end{aligned}
$$

From the condition of orthogonality (see Example 186, on p. 270) it follows

$$
\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = \begin{cases} 1, & \text{for } n = 1, \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\int_0^2 \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = \begin{cases} 1, & \text{for } n = 2, \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\int_0^2 \sin(2\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = \begin{cases} 1, & \text{for } n = 4, \\ 0, & \text{otherwise.} \end{cases}
$$

Finally, for constants b_n we obtain:

$$
\begin{aligned}
b_1 &= 2\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x - 0 + 0 = 2, \\
b_2 &= 0 - \int_0^2 \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x + 0 = -1, \\
b_4 &= 0 - 0 + 4\int_0^2 \sin(2\pi x) \sin\left(\frac{n\pi x}{2}\right) \mathrm{d}x = 4, \\
b_n &= 0 \quad \text{if} \quad n \neq 1, 2, 4,
\end{aligned}
$$

and the solution is

$$
u(x,t) = 2\sin\left(\frac{\pi x}{2}\right) \exp\left(-\frac{\pi^2}{16}t\right) - \sin(\pi x) \exp\left(-\frac{\pi^2}{4}t\right) - \sin(2\pi x)\, exp\left(-\pi^2 t\right).
$$

Problem 267

Find the solution of the following PDE

$$
u_t = \alpha^2 u_{xx}, \quad 0 \leq x \leq \pi,\ t > 0,
$$

if the boundary conditions are

$$u(0,t) = 0,$$
$$u_x(\pi,t) = 0,$$

and the initial conditions are

$$u(x,0) = 3\sin\left(\frac{5x}{2}\right) = f(x).$$

Solution

In this case we will use the method of separation of variables, i.e. we shall assume a solution in the form $u(x,t) = X(x)T(t)$, so that the initial PDE comes down to two ordinary differential equations

$$\frac{\dot{T}}{a^2 T} = \lambda \quad \wedge \quad \frac{X''}{X} = \lambda, \quad \lambda = \text{const.}$$

and the respective conditions

$$u(0,t) = X(0)\cdot T(t) = 0 \quad \Rightarrow \quad X(0) = 0,$$
$$u_x(\pi,t) = X'(\pi)\cdot T(t) = 0 \quad \Rightarrow \quad X'(\pi) = 0,$$
$$u(x,t) = X(x)\cdot T(0) = 3\sin\left(\frac{5x}{2}\right) \equiv \varphi(x).$$

Similarly, as in the previous case, from

$$X'' - \lambda X = 0, \quad \text{for } -\lambda = \mu^2, \quad \text{i.e.} \quad X'' + \mu^2 X = 0$$

we obtain

$$X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x).$$

Now, from boundary conditions, we obtain:

$$X(0) = 0 \quad \Rightarrow \quad C_2 = 0,$$
$$X'(\pi) = 0 \quad \Rightarrow \quad \mu C_1 \cos(\pi\mu) = 0 \quad \Rightarrow \quad \pi\mu = \frac{2n-1}{2}\pi,$$
$$\mu = \frac{2n-1}{2} = n - \frac{1}{2}, \quad n = 1,2,\ldots,$$
$$\lambda_n = -\left(n - \frac{1}{2}\right)^2,$$
$$X_n = \sin\left[\left(n - \frac{1}{2}\right)x\right].$$

For the function $T(t)$, we obtain

$$\frac{\mathrm{d}T}{T} = a^2\lambda \Big| \int \quad \Rightarrow \quad \ln T = a^2\lambda t \quad \Rightarrow$$
$$T = \exp\left[-a^2\left(n - \frac{1}{2}\right)^2 t\right],$$

and thus

$$u_n(x,t) = X_n(x)\,T_n(t) = \sin\left(\left(n-\frac{1}{2}\right)x\right)\exp\left(-\alpha^2\left(n-\frac{1}{2}\right)^2 t\right)$$

and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\left(n-\frac{1}{2}\right)x\right)\exp\left(-\alpha^2\left(n-\frac{1}{2}\right)^2 t\right).$$

The constants b_n are determined from the initial condition

$$u(x,0) = 3\sin\left(\frac{5x}{2}\right),$$

i.e.

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\left(n-\frac{1}{2}\right)x\right) = 3\sin\left(\frac{5x}{2}\right).$$

It is necessary to expand the function $\varphi(x)$ into a Fourier series, and thus

$$b_n = \frac{2}{\pi}\int_0^{\pi} 3\sin\left(\frac{5x}{2}\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\,\mathrm{d}x = \frac{6}{\pi}\cdot\begin{cases}\pi/2, & \text{for } n=3,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, only $b_3 \neq 0$, while other constants are equal to zero. Finally, the solution is

$$u(x,t) = 3\sin\left(\frac{5x}{2}\right)\exp\left[-\left(\frac{5\alpha}{2}\right)^2 t\right].$$

Check:

$$u_t(x,t) = -\frac{75}{4}\alpha^2\sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right),$$

$$u_{xx}(x,t) = -\frac{75}{4}\sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right),$$

and thus

$$u_t = \alpha^2 u_{xx},$$

i.e. the solution is correct.

Problem 268

Solve the PDE

$$u_t = u_{xx}, \quad 0 \leq x \leq 2\pi,\ t > 0.$$

The boundary conditions are

$$u_x(0,t) = 0,$$
$$u_x(2\pi,t) = 0,$$

and the initial condition is

$$u(x,0) = x = f(x).$$

Note. In this case, as in the several previous cases, we shall use the method of separation of variables, which leads to the problem of main values.
The first part of the solution is the same. The difference is only in boundary conditions, which lead to different solutions.

Solution

We start from the assumption that the solution can be represented in the form $u(x,t) = X(x)\,T(t)$, and by division by $T \cdot X$, we obtain

$$\frac{\dot{T}}{T} = \frac{X''}{X} = \lambda, \quad \lambda = \text{const}. \tag{7.321}$$

From boundary conditions we obtain:

$$u_x(0,t) = X'(0)\,T(t) = 0 \quad \Rightarrow \quad X'(0) = 0 \quad (\text{because } T(t) \text{ is not equal to zero for each } t),$$
$$u_x(\pi,t) = X'(2\pi)\,T(t) = 0 \quad \Rightarrow \quad X'(2\pi) = 0 \quad (\text{because } T(t) \text{ is not equal to zero for each } t). \tag{7.322}$$

From (7.321) it follows

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \lambda\,T(t) \quad \Rightarrow \quad \frac{\mathrm{d}T}{T} = \lambda\,\mathrm{d}t \quad \Rightarrow \quad T = C\,e^{\lambda\,t}. \tag{7.323}$$

Also, from (7.321) and the boundary conditions (7.322), we obtain

$$X'' - \lambda\,X = 0, \quad X = c\,e^{\alpha t} \quad \Rightarrow \quad X'' = c\,\alpha^2\,e^{\alpha t} \quad \Rightarrow$$
$$c\,e^{\alpha t}(\lambda^2 - 1) = 0 \quad \Rightarrow \quad \alpha = \pm\sqrt{\lambda},$$
$$X(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x), \quad \alpha = -\mu^2,$$

$$X'(0) = \alpha\,c_1 \cos 0 - \alpha\,c_2 \sin 0 = \alpha\,c_1 = 0 \quad \Rightarrow \quad c_1 = 0,$$
$$X'(2\pi) = \alpha\,\cancelto{0}{c_1} \cos 2\pi\alpha - \alpha\,c_2 \sin 2\pi\alpha = \alpha\,c_2 \sin 2\pi\alpha = 0.$$

As we have already emphasized, $c_2 \neq 0$ (non-trivial solution), and thus

$$\sin 2\pi\alpha = 0 \quad \Rightarrow \quad 2\pi\alpha = n\pi, \quad \Rightarrow \quad \alpha = \frac{n}{2}, \quad n = 1,2,\ldots$$

Thus, $\lambda_n = -\left(\frac{n}{2}\right)^2$, and it follows that

$$X_n(x) = a_n \cos\left(\frac{nx}{2}\right).$$

For λ_n determined in this way, from (7.323), for function $T(t)$ we obtain

$$T_n = b_n e^{-\frac{n^2}{4}t}$$

and thus

$$u_n(x,t) = X_n(x)\,T_n(t) = c_n \cos\left(\frac{nx}{2}\right) a_n \exp\left(-\frac{n^2 t}{4}\right).$$

Note that there is a solution also for $n = 0$, which we will denote by $u_0 = b_0 = a_0/2$.

The final solution (applying the principle of superposition) is of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{b_n \cdot c_n}_{a_n} \cos\left(\frac{nx}{2}\right) \exp\left(-\frac{n^2 t}{4}\right). \tag{7.324}$$

From the initial condition $u(x,0) = \varphi(x) = x$ we obtain the equation for determining the constants a_n

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right) = x,$$

where Euler coefficients of the Fourier series are

$$a_n = \frac{2}{L}\int_0^L \varphi(x)\cos\left(\frac{nx}{2}\right)\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x\cos\left(\frac{nx}{2}\right)\mathrm{d}x.$$

By partial integration ($\int u\,v = uv - \int v\,\mathrm{d}u$, $u = x$, $\mathrm{d}v \cos\frac{nx}{2}\,\mathrm{d}x$) we obtain

$$a_n = \frac{1}{\pi}\left[\frac{2x}{n}\sin\left(\frac{nx}{2}\right)\Big|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi}\sin\left(\frac{2}{n}\right)\mathrm{d}x\right] =$$

$$= \left[-\frac{2}{n}\left(-\frac{2}{n}\right)\cos\left(\frac{nx}{2}\right)\Big|_0^{2\pi}\right] = \frac{4}{\pi n^2}\left(\cos(n\pi) - 1\right).$$

Given that $\cos(n\pi) = (-1)^n$, for a_n we obtain

$$a_n = \frac{4}{\pi n^2}\left[(-1)^n - 1\right].$$

Note that even coefficients are $(n = 2m)$ $a_{2m} = 0$, because $(-1)^n - 1 = 1 - 1 = 0$, and odd coefficients are $(n = 2m-1)$ $a_{2m-1} = -\frac{8}{\pi(2m-1)^2}$, because $(-1)^{2m-1} - 1 = -2$.

Separating even and odd members

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right)\exp\left(-\frac{n^2 t}{4}\right) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2m}\cos(mx)\exp(-m^2 t) + \sum_{n=1}^{\infty} a_{2m-1}\cos\left(\frac{(2m-1)}{2}x\right)\exp\left(-\frac{(2m-1)^2 t}{4}\right),$$

bearing in mind that the even members $a_{2m} = 0$, finally we obtain

$$u(x,t) = \pi - \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2m-1)}{2}x\right)\exp\left(-\frac{(2m-1)^2 t}{4}\right).$$

Note, finally, that

$$\lim_{t\to\infty u(x,t)} = \pi.$$

Problem 269

Solve the equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{7.325}$$

with the conditions

$$u(x,0) = x - x^2, \quad u_x(0,t) = u_x(1,t) = 0. \tag{7.326}$$

Solution

Starting with the substitution $u(x,t) = X(x) \cdot T(t)$ and dividing the initial equation (7.325), we obtain

$$\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda$$

that is

$$\dot{T} = -\lambda T \quad \wedge \quad X'' = -\lambda X.$$

Integration of the first equation yields

$$T = e^{-\lambda t}. \tag{7.327}$$

Note that we did not include here the integration constant, as it is "absorbed" by constants that appear later (because $X \cdot T$, see 7.324).

From

$$X'' + \lambda X = 0 \quad \Rightarrow$$
$$X(x) = c_1 \sin kx + c_2 \cos kx,$$

and from the initial conditions ($X'(0) = X'(1) = 0$) we obtain

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0, \tag{7.328}$$

which leads to the equation

$$c_1 = 0, \quad \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots, n\pi.$$

Substituting into (7.327), for T we obtain

$$T_n(t) = c_3 e^{-n^2\pi^2 t}$$

and

$$X_n(x) = c_2 \cos(n\pi x),$$

and thus

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos(n\pi x), \quad (a_n \equiv c_1 c_3).$$

in order to determine the constants a_n we shall use the initial conditions, expanding the given functions $(x - x^2)$ into Fourier series

$$u(x,0) = x - x^2 = \sum_{n=0}^{\infty} a_n \cos(n\pi x) =$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

From here we obtain:

$$a_0 = 2\int_0^1 \left(x - x^2\right)\,\mathrm{d}x = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Big|_0^1 = \frac{1}{3}, \tag{7.329}$$

$$a_n = 2\int_0^1 \left(x - x^2\right)\cos(n\pi x)\,\mathrm{d}x = \tag{7.330}$$

$$= 2\,\frac{1-2x}{n^2\pi^2}\sin(n\pi x) - \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right)\sin(n\pi x)\Big|_0^1 = \tag{7.331}$$

$$= -\frac{2}{n^3\pi^3}\left(1 + (-1)^n\right). \tag{7.332}$$

Thus, the solution of the PDE is

$$u(x,t) = \frac{1}{3} + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)(n+1)-1}{n^3}\,e^{-n^2\pi^2 t}\cos(n\pi x). \tag{7.333}$$

Problem 270

Find the solution for the PDE

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0, \tag{7.334}$$

that satisfies the following conditions

$$u(x,0) = \begin{cases} x, & \text{for } 0 < x < 1, \\ 2 - x, & \text{for } 1 < x < 2. \end{cases}, \quad u(0,t) = u_x(2,t) = 0. \tag{7.335}$$

Solution

As in previous problems, we start from the assumption

$$u(x,t) = X(x)\,T(t), \tag{7.336}$$

which leads to a system of ordinary DE

$$\dot{T} = \lambda\,T, \quad X'' = \lambda\,X.$$

The solution of the first DE is

$$T = c\,e^{\lambda t}, \quad \lambda = -k^2, \tag{7.337}$$

while the solution of the second differential equation is

$$X(x) = c_1 \sin kx + c_2 \cos kx$$

boundary conditions (7.335), for function X, are

$$u(0,t) = X(0)\cdot T(t) = 0 \quad \Rightarrow \quad X(0) = 0,$$

$$u_x(2,t) = X'(2)\cdot T(t) = 0 \quad \Rightarrow \quad X'(2) = 0,$$

from where it follows

$$X(0) = c_1 \sin 0 + c_2 \cos 0 = 0 \quad \Rightarrow \quad c_2 = 0,$$

$$X'(2) = c_1 k \cos 2k + \overset{0}{\cancel{c_2}} k \sin 2x = 0 \quad \Rightarrow \quad \cos 2k = 0$$

and we further obtain

$$k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots, \frac{(2n-1)\pi}{4}.$$

Substituting the value k, into (7.337), we obtain

$$T_n(t) = c_3 e^{-\frac{(2n-1)^2}{16}\pi^2 t},$$

that is, the solution

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{b_n}_{c_1 \cdot c_3} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{(2n-1)}{4}\pi x.$$

The constants b_n are determined from the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)}{4}\pi x = \varphi(x) = \begin{cases} x, & \text{for } 0 < x < 1, \\ 2-x, & \text{for } 1 < x < 2. \end{cases}.$$

Using the expansion of function $\varphi(x)$ into a Fourier series, for constants b_n we obtain:

$$\begin{aligned} b_n &= 2\int_0^1 x \sin \frac{(2n-1)}{4}\pi x \, \mathrm{d}x + \int_1^2 (2-x) \sin \frac{(2n-1)}{4}\pi x \, \mathrm{d}x = \\ &= \left[\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x - \frac{8x}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_0^1 + \\ &+ \left[-\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x + \frac{8(x-2)}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_1^2 = \\ &= \frac{32}{(2n-1)^2\pi^2} \left(\sin \frac{(2n-1)\pi}{4} + \cos(n\pi) \right), \end{aligned}$$

and the solution of the initial PDE is

$$u(x,t) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{(2n-1)\pi}{4} + \cos(n\pi) \right)}{(2n-1)^2} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{2n-1}{4}\pi x \qquad (7.338)$$

Problem 271

Find the solution for the PDE

$$u_t = u_{xx}, \quad 0 < x < 3, \quad t > 0 \qquad (7.339)$$

that satisfies the following conditions

- initial

$$u(x,0) = 4x - x^2, \qquad (7.340)$$

- boundary

$$u(0,t) = 0, \quad u(3,t) = 3. \tag{7.341}$$

Solution

Let us look for the solution of the problem by applying the method of separation of variables, i.e. $u(x,t) = X(x)T(t)$. Boundary conditions, for function $X(x)$ yield

$$u(0,t) = X(0) \cdot T(t) = 0 \quad \Rightarrow \quad X(0) = 0,$$
$$u(3,t) = X(3) \cdot T(t) = 3.$$

The second condition does not separate the boundary conditions, and it is thus not suitable for solving by this method.

The question arises: is it possible to find a transformation

$$u(x,t) = f(v(x,t))$$

to separate both the PDE and the conditions?

Let us try with a linear transformation

$$u(x,t) = v(x,t) + ax + b.$$

The constants a and b should be determined so that the initial PDE is transformed into an equation of the same form, i.e. $v_t = v_{xx}$, and that the conditions are separated, i.e. in our case: $v(0,t) = 0$ and $v(3,t) = 0$.

In our case we obtain:

$$u_{xx} = v_{xx} + 0 + 0 = v_{xx}, \quad u_t = v_t + 0 + 0 = v_t \quad \Rightarrow$$
$$v_{xx} = v_t.$$

The boundary conditions are

$$u(0,t) = v(0,t) + a \cdot 0 + b = 0 \quad \Rightarrow \quad v(0,t) = b = 0,$$
$$u(3,t) = v(3,t) + a \cdot 3 + b = 0 \quad \Rightarrow \quad v(0,t) = 3 + 3a = 0 \quad \Rightarrow \quad a = -1,$$
$$u = v + x,$$
$$v(0,t) = 0,$$
$$v(3,t) = 0.$$

The former shows that it is possible to find a suitable transformation!

The problem now comes down to solving the equation

$$v_t = v_{xx}$$

with boundary conditions

$$v(0,t) = 0, \quad v(3,t) = 0, \tag{7.342}$$

and initial condition

$$v(x,0) = u(x,0) - x = 3x - x^2.$$

Let us now apply the method of separation $X(x) \cdot T(t)$:

$$v_t = v_{xx} \quad \Rightarrow \quad X^{''} \cdot T = \dot{T} \cdot X = 0.$$

From here, dividing by XT, we obtain

$$\frac{X^{''}}{X} = \frac{\dot{T}}{T} = -k^2,$$

that is a system of two ordinary DE:

$$X^{''} + k^2 X = 0 \quad \wedge \quad \dot{T} + k^2 T = 0.$$

The boundary conditions for the first equation are given by equations (7.342).

The solution of the first equation is of the form (as already demonstrated several times)

$$X(x = c_1 \sin kx + c_2 \cos kx.)$$

From (7.342) we obtain:

$$\begin{aligned} X(0) &= c_2 = 0, \\ X(3) &= c_1 \sin 3x = 0 \quad \Rightarrow \quad k = \frac{n\pi}{3}, \\ X &= c_1 \sin \frac{n\pi}{3} x. \end{aligned}$$

From the second of equation we obtain

$$\dot{T} + k^2 T = 0 \quad T = c_3 e^{-\frac{n^2\pi^2}{9}t}.$$

Note that these solutions depend on $n = 1, 2, \ldots$, and thus on of the solutions is

$$v_n = X_n \cdot T_n = \underbrace{c_1 c_3}_{b_n} e^{-\frac{n^2\pi^2}{9}t} \cdot \sin \frac{n\pi}{3} x.$$

Applying the principle of superposition, we obtain

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x.$$

The constants b_n are obtained from the initial condition

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{3} x = 3x - x^2$$

expanding the function $3x - x^2$ into the Fourier sine series, i.e.

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 (3x - x^2) \sin \frac{n\pi}{3} x \, dx = \\ &= \left[-\frac{6(2x-3)}{n^2\pi^2} \sin \frac{n\pi}{3} x + \frac{3(n^2\pi^2 x^2 - 3n^2\pi^2 x - 18)}{n^3\pi^3} \cos \frac{n\pi}{3} x \right] \Bigg|_0^3 = \\ &= \frac{32(1 - (-1)^n)}{n^3\pi^3}. \end{aligned}$$

Finally, for function $v(x,t)$, we obtain

$$v(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}e^{-\frac{\pi^2 n^2}{9}t}\sin\frac{n\pi}{3}x,$$

and the solution of the initial equation is $u = v + x$,

$$u(x,t) = x + \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}e^{-\frac{\pi^2 n^2}{9}t}\sin\frac{n\pi}{3}x. \tag{7.343}$$

R In this case also an analysis of the expression $1-(-1)^n$ can be performed, from which it can be seen that the even members are equal to zero, and that the function u is expressed in terms of odd members only, which we leave to the reader.

Problem 272

Solve the PDE

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{7.344}$$

with the conditions

$$\underbrace{u(x,0) = 0,}_{\text{initial}} \quad \underbrace{u_x(0,t) = -1, \quad u_x(1,t) = 0}_{\text{boundary}}. \tag{7.345}$$

Solution

The problem is, conceptually, similar to the previous, but in this case the substitution $u = v + ax + b$ does not lead to suitable boundary conditions, because $u_x = v_x + a$, and thus

$$\begin{aligned} u_x(0,t) &= v_x(0,t) + a = -1 \quad \Rightarrow \quad a = -1, \quad (\text{for } v_x(0,t) = 0) \\ u_x(1,t) &= v_x(1,t) + a = 0 \quad \Rightarrow \quad a = 0 \quad (\text{for } v_x(1,t) = 0) \end{aligned}$$

which is contradictory.

We shall try with $u = v + ax^2 + bx$.

This yields

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx} + 2a, \tag{7.346}$$

and we do not obtain the same partial equation.

Let us try the following transformation

$$u = v + a(x^2 + 2t) + bx \tag{7.347}$$

which yields the equation

$$\left.\begin{aligned} u_{xx} &= v_{xx} + 2a, \\ u_t &= v_t + 2a \end{aligned}\right\} \quad \Rightarrow \quad v_{xx} = v_t. \tag{7.348}$$

Thus, the equation is the same, for $u(x,t) = v(x,t) + \frac{1}{2}(x^2+2t) - x$.

It remains to determine the boundary conditions

$$\begin{aligned} u_x(0,t) &= v_x(0,t)2 + ax_{x=0} + b = -1 \quad \Rightarrow \quad b = -1, \\ u_x(1,t) &= v_x(1,t)_2 ax_{x=1} + b = -1 \quad \Rightarrow \quad a = \frac{1}{2}. \end{aligned} \tag{7.349}$$

The initial condition becomes

$$v(x,0) = x - \frac{1}{2}x^2. \tag{7.350}$$

Separating the variables $v = XT$ yields the equations $X'' = -k^2 X$ and $T' = -k^2 T$ from where we obtain

$$X = c_1 \sin kx + c_2 \cos kx, \quad X' = c_1 \cos kx - c_2 \sin kx, \tag{7.351}$$

while the boundary conditions (7.349) yield

$$c_1 = 0, \quad k = n\pi, \tag{7.352}$$

and thus

$$X_n(x) = c_2 \cos n\pi x, \tag{7.353}$$

$$T_n(t) = c_3 e^{-n^2\pi^2 t}, \tag{7.354}$$

and

$$v_n(x,t) = c_2 c_3 e^{-n^2\pi^2 t} \cos n\pi x.$$

Note that $T_0 = c_3$ and $X_0 = c_2$, so a constant exists for the member $X_0 \cdot T_0 = c_2 c_3$.

Finally, using the principle of superposition, we obtain for $v(x,t)$

$$v(x,t) = \underbrace{c_2 c_3}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x.$$

The constants a_n are determined from the initial condition (7.350), by expanding the known function $x - \frac{1}{2}x^2$ into a Fourier cosine series, i.e.

$$\begin{aligned} a_0 &= \frac{2}{1}\int_0^1 \left(x - \frac{1}{2}x^2\right) \mathrm{d}x = x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{2}{3}, \\ a_n &= \frac{2}{1}\int_0^1 \left(x - \frac{1}{2}x^2\right) \cos n\pi x \,\mathrm{d}x = \\ &= \frac{-2(x-1)}{n^2\pi^2} \cos n\pi x - \left(\frac{(2x-x^2)}{n\pi} + \frac{2}{n^3\pi^3}\right) \sin n\pi x \Big|_0^1 = \\ &= -\frac{2}{n^2\pi^2}, \end{aligned}$$

and thus v is of the form

$$v(x,t) = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x$$

and the solution of the initial PDE is

$$u(x,t) = \frac{1}{2}(x^2+2t) - x + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x \tag{7.355}$$

R From the last two problems the question arises, is it possible to transform the PDE

$$u_t = u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{7.356}$$

with conditions

$$u(x,0) = f(x), \quad u(0,t) = p(t), \quad u(L,t) = q(t), \tag{7.357}$$

so that we obtain the standard boundary conditions ($X(a) = 0$, $X(b) = 0$)? The answer is YES.
The transformation is

$$u = v + A(t)x + B(t) \tag{7.358}$$

and the conditions are

$$u(L,t) = \underbrace{v(L,t)}_{=0} + A(t)\cdot L + B(t) \quad \Rightarrow \quad u(L,t) = A(t)\cdot L + B(t) = q(t),$$
$$u(0,t) = \underbrace{v(0,t)}_{=0} + A(t)\cdot 0 + B(t) \quad \Rightarrow \quad B(t) = p(t).$$

From these equations it follows that $A(t) = \dfrac{q-p}{L}$.
The partial differential equation, for v is

$$\dot{v} = v_{xx} - \dot{q}(t)\,\frac{x}{L} + \dot{p}\,\frac{(L-x)}{L},$$

and the initial condition

$$v(x,0) = f(x) + \frac{p(0)-q(0)}{L}x - p(0).$$

Problem 273

Determine the PDE type and find its solution (conduction of heat equation - with source)

$$u_t = u_{xx} + Q(x), \quad 0 < x < L, \quad t > 0, \tag{7.359}$$

that satisfies the following conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x). \tag{7.360}$$

Solution

When solving this problem, we will first analyze the special case when $L = 2$, $f(x) = 2x - x^2$ and $Q(x) = 1 - |x-1|$. If we observe this problem without the source, the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\,\sin\frac{n\pi x}{2}, \tag{7.361}$$

where

$$T_n(t) = \frac{16(1-(-1)^n)}{n^3\pi^3}\,e^{-\frac{n^2\pi^2}{4}t} \tag{7.362}$$

The function $Q(x)$ should now be represented by a Fourier sine series, i.e.

$$Q(x) = \sum_{n=1}^{\infty} q_n\,\sin\frac{n\pi x}{2}, \tag{7.363}$$

which in the given special case for Q

$$Q(x) = \begin{cases} x, & \text{if } 0 < x < 1, \\ 2-x, & \text{if } 1 < x < 2. \end{cases} \tag{7.364}$$

yields for q_n

$$\begin{aligned} q_n &= \int_0^2 Q(x) \sin \frac{n\pi x}{2} \, \mathrm{d}x = \\ &= \int_0^1 x \sin \frac{n\pi x}{2} \, \mathrm{d}x + \int_1^2 (2-x) \sin \frac{n\pi x}{2} \, \mathrm{d}x = \\ &= \left[\frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} - 2xn\pi \cos \frac{n\pi x}{2} \right] \Bigg|_0^1 + \\ &+ \left[-\frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} + \frac{2x}{n\pi} \cos \frac{n\pi x}{2} \right] \Bigg|_1^2 = \\ &= \frac{8}{n^2\pi^2} \sin \frac{n\pi x}{2}. \end{aligned} \tag{7.365}$$

Substituting the respective expressions into (7.359), we obtain

$$\sum_{n=1}^{\infty} T_n'(t) \sin \frac{n\pi x}{2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi x}{2} \right)^2 T_n(t) \sin \frac{n\pi x}{2} + \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{2},$$

By simplifying and grouping the respective members next to $\sin \frac{npx}{2}$, we obtain

$$T_n'(t) + \frac{n^2\pi^2}{4} T_n(t) = q_n, \tag{7.366}$$

which is a linear differential equation for T_n. Its solution is

$$T_n(t) = \frac{4}{n^2 \pi^2} q_n + b_n e^{-(\frac{n\pi}{2})^2 t}$$

The final solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2 \pi^2} q_n + b_n e^{-(\frac{n\pi}{2})^2 t} \right) \sin \frac{n\pi x}{2}. \tag{7.367}$$

The initial condition yields

$$2x - x^2 = \sum_{n=1}^{\infty} \left(\frac{4}{n^2 \pi^2} q_n + b_n \right) \sin \frac{n\pi x}{2}.$$

If we introduce the substitution

$$c_n = \frac{4}{n^2 \pi^2} q_n + b_n, \tag{7.368}$$

we obtain

$$2x - x^2 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2}.$$

This expression represents the Fourier sine series, and thus c_n

$$\begin{aligned} c_n &= \int_0^2 (2x - x^2) \sin \frac{n\pi x}{2} \, \mathrm{d}x = \\ &= \frac{16}{n^3\pi^3} \left(1 - \cos \frac{n\pi}{2} \right), \end{aligned} \tag{7.369}$$

and b_n

$$b_n = c_n - \frac{4}{n^2\pi^2} q_n.$$

The final solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + \left(c_n - \frac{4}{n^2\pi^2} q_n \right) e^{-(\frac{n\pi}{2})^2 t} \right) \sin \frac{n\pi x}{2} \tag{7.370}$$

where q_n and c_n are determined by (7.365) and (7.368), respectively.

Problem 274

Find the equation for the motion of points on a circular membrane.

Solution

In this case we will express the Delta operator in terms of polar coordinates, and thus the equation (7.13) becomes

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{7.371}$$

The unknown function u is $u = u(r, \varphi, t)$. In the case of radial symmetry u does not depend on φ, and thus $u = u(r,t)$. For this case the initial equation (7.13) obtains the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right). \tag{7.372}$$

Assume that the membrane is fixed on its border (from where the boundary conditions follow), i.e.

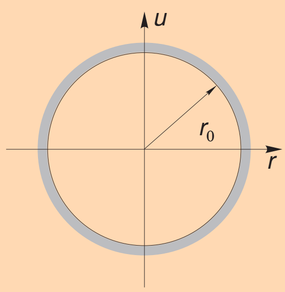

Figure 7.6: Circular membrane.

$$u(r_0, t) = 0, \quad \text{for } t \geq 0. \tag{7.373}$$

Here r_0 represents the radius of the circular membrane. In addition, assume that

– the initial value of the motion (initial condition) is

$$u(r,0) = f(r) \quad \text{and} \tag{7.374}$$

– the initial velocity (second initial condition, pertaining to the derivative of the variable)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r). \tag{7.375}$$

Let us now apply the method of separation of variables, starting from the assumption that

$$u(r,t) = R(r)T(t). \tag{7.376}$$

Under this condition, the initial equation (7.372) is decomposed to two ordinary differential equations

$$R'' + \frac{1}{r}R' + k^2 R = 0, \tag{7.377}$$

$$\ddot{T} + c^2 k^2 G = 0, \tag{7.378}$$

where k is a constant, undetermined for now. In previous relations, a point above a variable denotes its derivative by the time t, and $()'$ the derivative by variable r. We have also introduced a new constant $\lambda^2 = c^2 k^2$.

Observe now the first equation. Let us introduce the substitution $s = kr$ (s is the new variable), where

$$R' = \frac{\mathrm{d}R}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = k\frac{\mathrm{d}R}{\mathrm{d}s}, \tag{7.379}$$

$$R'' = \frac{\mathrm{d}}{\mathrm{d}r}\left(k\frac{\mathrm{d}R}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(k\frac{\mathrm{d}R}{\mathrm{d}s}\right)\frac{\mathrm{d}s}{\mathrm{d}r} = k^2\frac{\mathrm{d}^2R}{\mathrm{d}s^2}. \tag{7.380}$$

By substituting these relations into equation (7.377) we obtain

$$\frac{\mathrm{d}^2R}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}R}{\mathrm{d}s} + R = 0. \tag{7.381}$$

This equation represents the Bessel equation (5.45) (for $\nu = 0$), whose general solution (5.98) is

$$R = C_1 J_0(s) + C_2 Y_0(s). \tag{7.382}$$

Here J_0 is a Bessel function is of first type zero order, and Y_0 a Bessel function of second type zero order.

Given that the displacement of the points of the membrane u is always finite, $Y_0 \to \infty$ when $s \to 0 \quad \Rightarrow \quad C_2 = 0$. It is obvious that $C_1 \neq 0$, as otherwise $R \equiv 0$, which would be a trivial solution. Without losing generality, we shall assume that $C_1 = 1$, from where we obtain

$$R = J_0(s) = J_0(kr). \tag{7.383}$$

On the border $r = r_0$ we have

$$u(r_0,t) = R(r_0)\cdot T(t) = 0 \quad \Rightarrow \quad R(r_0) = 0 \quad \Rightarrow \quad J_0(kr_0) = 0. \tag{7.384}$$

The Bessel function J_0 has an infinite number of real zeros $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$, and thus, from equation $J_0(kr_0) = 0$ it follows that

$$\alpha_m = kr_0 \quad \Rightarrow \quad k = \frac{\alpha_m}{r_0} = k_m, \quad m = 1,2,\ldots \tag{7.385}$$

Now, substituting this relation, we obtain

$$R_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{r_0} r\right). \tag{7.386}$$

Given that

$$\lambda^2 = c^2 k^2 = c^2 k_m^2 = \lambda_m^2, \tag{7.387}$$

the equation (7.378) becomes

$$\ddot{T} + \lambda_m^2 T = 0 \quad \Rightarrow \quad T_m(t) = a_m \cos\lambda_m t + b_m \sin\lambda_m t, \tag{7.388}$$

From where we obtain

$$u_m = T_m(t) R_m(r) = (a_m \cos\lambda_m t + b_m \sin\lambda_m t)\, J_0\left(\frac{\alpha_m}{r_0} r\right). \tag{7.389}$$

From here we further obtain for solution u

$$u = \sum_{m=1}^{\infty} u_m = \sum_{m=1}^{\infty} (a_m \cos\lambda_m t + b_m \sin\lambda_m t)\, J_0\left(\frac{\alpha_m}{r_0} r\right). \tag{7.390}$$

Furthermore, the constants a_m and b_m must be determined. For determining them, we have the unused initial conditions available. From the first condition (7.374) we obtain

$$u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{r_0} r\right) = f(r), \tag{7.391}$$

from where we can determine the coefficients a_m as well as the coefficients of the Fourier-Bessel series, which represents the expansion of the known function $f(r)$, and thus we obtain

$$a_m = \frac{2}{r_0^2 J_1^2(\alpha_m)} \int_0^{r_0} r f(r) J_0\left(\frac{\alpha_m}{r_0} r\right) \mathrm{d}r, \quad m = 1, 2, \ldots \tag{7.392}$$

Similarly, we obtain also the coefficients b_m from the remaining condition (7.375)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r). \tag{7.393}$$

Problem 275

The electric potential of a point source in a homogeneous isotropic medium satisfies the Laplace equation, which in the cylindrical coordinate system has the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0.$$

Solve this equation, with two boundary conditions

- the potential along the boundary planes of the medium is constant, and
- the current flux in the direction perpendicular to the boundary plane is constant.

Solution

In this case we will look for the solution using the method of separation of variables

$$U = R(r)Z(z).$$

By substitution in the initial equation we obtain two ordinary differential equations

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -m^2, \quad \frac{Z''}{Z} = m^2, \tag{7.394}$$

where m^2 is a constant. The second equation is a second order homogeneous differential equation with constant coefficients, whose solution is

$$Z(z) = C_1 e^{-mz} + C_2 e^{+mz}.$$

The first equation is a special form of Bessel equation of zero order, whose solutions are Bessel functions of the first and second type, of zero order. Bessel functions of the second type do not correspond to the nature of electricity potential of a point source, which tends to zero in infinity, and thus the solution of this equation will be of the form

$$R(r) = J_0(mr).$$

The final solution is

$$U(r,z) = \left[C_1 e^{-mz} + C_2 e^{+mz}\right] J_0(mr). \tag{7.395}$$

As the constant m cannot take all values, and given the fact that it was introduced in the expression (7.394) as m^2, its change in the interval $(0,\infty)$ should be observed, so the most general form of the solution (7.395) will be obtained by integrating for parameter m

$$U(r,z) = \int_0^\infty \left[C_1 e^{-mz} + C_2 e^{+mz}\right] J_0(mr)\,\mathrm{d}m.$$

For determining the constants C_i the boundary conditions are used.

Problem 276

Determine the motion of the points on a rectangular membrane.

Solution

In this case we will represent the initial equation (7.13) in Cartesian coordinates, thus

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7.396}$$

with boundary conditions

$$u = 0, \quad \text{(at the membrane borders for } \forall t \geq 0) \tag{7.397}$$

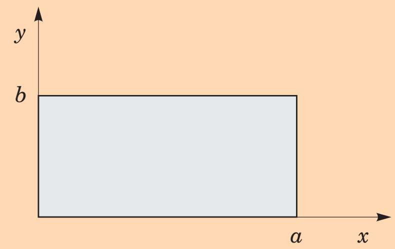

Figure 7.7: Rectangular membrane.

and initial conditions

$$u(x,y,0) = f(x,y), \tag{7.398}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x,y). \tag{7.399}$$

In this case we will look for the solution by applying the Fourier method

$$u(x,y,t) = F(x,y)T(t), \tag{7.400}$$

and thus the equation (7.396) is decomposed on two

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{1}{F}(F_{xx}+F_{yy}) = -\nu^2 \quad \Rightarrow$$

$$\ddot{T} + \lambda^2 T = 0 \quad \text{i} \tag{7.401}$$

$$F_{xx} + F_{yy} + \nu^2 F = 0. \tag{7.402}$$

We have introduced here the notation $c^2\nu^2 = \lambda^2$. Further, assume the equation (7.402) can also be represented in the from

$$F(x,y) = X(x)Y(y). \tag{7.403}$$

With this assumption, the equation (7.402) becomes

$$Y(y)X_{xx}(x) + X(x)Y_{yy}(y) + \nu^2 XY = 0 \quad \Rightarrow \tag{7.404}$$

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\frac{1}{Y}\left(\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \nu^2 Y\right) = -k^2. \tag{7.405}$$

From here we obtain:

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + k^2X = 0, \tag{7.406}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + (\nu^2 - k^2)Y = 0. \tag{7.407}$$

The solutions of this system are

$$X'' + k^2X = 0 \quad \Rightarrow \quad X = A\cos kx + B\sin kx, \tag{7.408}$$

$$Y'' + p^2Y = 0 \quad \Rightarrow \quad Y = C\cos py + D\sin py. \tag{7.409}$$

The boundary conditions (7.398) become

$$
\begin{aligned}
&u(x,y,t) = X(x)Y(y)T(t) \quad \Rightarrow \\
&u|_{x=0} = X(0)Y(y)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 \quad \Rightarrow \quad A = 0, \\
&u|_{x=a} = X(a)Y(y)T(t) = 0 \quad \Rightarrow \quad X(a) = 0 \quad \Rightarrow \quad B\sin kt = 0 \quad \Rightarrow \\
&\Rightarrow \quad k_m a = m\pi, \\
&u|_{y=0} = X(x)Y(0)T(t) = 0 \quad \Rightarrow \quad Y(0) = 0 \quad \Rightarrow \quad C = 0, \\
&u|_{y=b} = X(x)Y(b)T(t) = 0 \quad \Rightarrow \quad Y(b) = 0 \quad \Rightarrow \quad D\sin pt = 0 \quad \Rightarrow \\
&\Rightarrow \quad p_n b = n\pi.
\end{aligned}
$$

Without losing in generality we can assume that $B = 1$ and $D = 1$, and thus we obtain:

$$X(x) = \sin\frac{m\pi x}{a} \quad \wedge \quad Y(y) = \sin\frac{n\pi y}{b}. \tag{7.410}$$

Now, for function F we obtain

$$F_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{7.411}$$

Given that

$$
\begin{aligned}
p^2 = \nu^2 - k^2, \quad \lambda = c\nu = c\sqrt{p^2 + k^2} \quad \Rightarrow \\
\lambda_{mn} c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \Rightarrow \\
\lambda_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}.
\end{aligned} \tag{7.412}
$$

Now the solution of equation (7.401) is

$$T_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t, \tag{7.413}$$

and for $u_{mn}(x,y,t)$ we obtain

$$u_{mn}(x,y,t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{7.414}$$

that is, for the total solution u

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{7.415}$$

Thus, the solution was obtained in the form of a double Fourier series. The coefficients are determined from the initial conditions (7.398) and (7.399)

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y). \tag{7.416}$$

If we introduce the substitution

$$K_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}, \tag{7.417}$$

we obtain

$$f(x,y) = \sum_{m=1}^{\infty} K_m(y) \sin \frac{m\pi x}{a}, \tag{7.418}$$

and thus for fixed y we obtain

$$K_m(y) = \frac{2}{a} \int_0^a f(x,y) \sin \frac{m\pi x}{a} \mathrm{d}x. \tag{7.419}$$

Thus, we have

$$B_{mn} = \frac{2}{b} \int_0^b K_m(y) \sin \frac{n\pi y}{b} \mathrm{d}y, \tag{7.420}$$

which yields the generalized Fourier formula

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \mathrm{d}x\mathrm{d}y. \tag{7.421}$$

From the second condition we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B^*_{mn} \lambda_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = g(x,y) \quad \Rightarrow \tag{7.422}$$

$$B^*_{mn} = \frac{4}{ab} \int_0^b \int_0^a g(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \mathrm{d}x\mathrm{d}y. \tag{7.423}$$

Problem 277

Solve the wave equation

$$\Delta\psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{7.424}$$

where the unknown function ψ is a function of a set of variables $(\mathbf{r}, t)$, a $\mathbf{r} = \mathbf{r}(r, \varphi, \theta)$. r, φ and θ are spherical coordinates, t is time, and c a constant.

Solution

We will look for the solution applying the method of separation of variables, assuming that it has the form

$$\psi(\mathbf{r}, t) = P(\mathbf{r}) \cdot T(t). \tag{7.425}$$

Starting from (7.425), the equation (7.424) becomes

$$T \cdot \Delta P - \frac{1}{c^2} P \cdot \ddot{T} = 0,$$

that is

$$\frac{\Delta P}{P} = \frac{1}{c^2} \frac{\ddot{T}}{T}.$$

Given that left hand side is a function of **r** only, and the right hand side of t, we can conclude that these expressions have to be constant, i.e.

$$\frac{\Delta P}{P} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -k^2. \tag{7.426}$$

Thus, the initial partial differential equation (7.424), by applying this method, is decomposed on two differential equations (7.426):

$$\ddot{T} + \omega^2 T = 0, \quad (\omega = kc), \tag{7.427}$$

$$\Delta P + k^2 P = 0. \tag{7.428}$$

The solution of the first of equation (7.427) (second order differential equation with constant coefficients) is of the form

$$T = C_1 \sin\omega t + C_2 \cos\omega t. \tag{7.429}$$

Observe now the equation (7.428). As we have assumed that **r** is a function of spherical coordinates, this equation can be represented in the form (see the form of Delta operator in spherical coordinates)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 P}{\partial\varphi^2} + k^2 P = 0. \tag{7.430}$$

For solving this partial differential equation we shall, once again, use the method of separation of variables, i.e. assume that the function P is a function of the form

$$P = R(r)\cdot\Theta(\theta)\cdot\Phi(\varphi). \tag{7.431}$$

Substituting (7.431) into (7.430) and dividing by $R\Theta\Phi$ we obtain

$$\frac{1}{r^2 R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{r^2\Phi\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + k^2 = 0. \tag{7.432}$$

If we multiply this equation by $r^2\sin^2\theta$, we can notice that the member $\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}$ depends on φ only, while the remaining members depend on r and θ. Based on this, as in the previous similar case, we conclude that

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -m^2, \tag{7.433}$$

$$-\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - k^2 r^2\sin^2\theta = -m^2, \tag{7.434}$$

where m is a constant.

From (7.433) it follows that the function Φ is of the form

$$\Phi = C_3 \sin m\varphi + C_4 \cos m\varphi. \tag{7.435}$$

Substituting (7.435) into (7.432) we obtain

$$\frac{1}{r^2 R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{r^2\sin^2\theta}(-m^2) + k^2 = 0.$$

Multiplying the last relation by r^2 we obtain two members, one is the function of r only, and the other of θ only

$$\left[\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+k^2r^2\right]+\left[\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{m^2}{\sin^2\theta}\right]=0.$$

This equation is now decomposed on two ordinary differential equations

$$\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{m^2}{\sin^2\theta}=const.=-l(l+1) \tag{7.436}$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+k^2r^2=const.=l(l+1). \tag{7.437}$$

Te first equation (7.436), after substitution $\cos\theta=z$, comes down to Legendre equation (5.21)

$$(1-z^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}z^2}-2z\frac{\mathrm{d}\Theta}{\mathrm{d}z}+\left[l(l+1)-\frac{m^2}{1-z^2}\right]\Theta=0,$$

the solution of which has the form

$$\Theta=C_5P_l^m(z)+C_6Q_l^m(z). \tag{7.438}$$

The second equation (7.437), after introducing a new function $u=R\sqrt{r}$, becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}+\left[k^2-\frac{(l+1/2)^2}{r^2}\right]u=0,$$

which is the Bessel equation (5.45), whose solution is of the form

$$u=R\sqrt{r}=C_7J_{l+1/2}(kr)+C_8Y_{l+1/2}(kr). \tag{7.439}$$

Based on (7.438), (7.435) and (7.439) we can now write the solution of the initial equation(7.424)

$$\psi=(C_1\sin\omega t+C_2\cos\omega t)\cdot(C_3\sin m\varphi+C_4\cos m\varphi)\cdot$$
$$\cdot[C_5P_l^m(\cos\theta)+C_6Q_l^m(\cos\theta)]\cdot\frac{1}{\sqrt{r}}\left[C_7J_{l+1/2}(kr)+C_8Y_{l+1/2}(kr)\right]. \tag{7.440}$$

The constants C_i, $(i=1,2,\dots,8)$ are determined from initial and boundary conditions.

Problem 278

Laplace equation, in 2D space, is of the form

$$u_{xx} + u_{yy} = 0. \tag{7.441}$$

We will show that it can also be solved by the method of separating variables.

Let us find the solution of this equation (7.441), if the boundary conditions are

$$u(x,0) = 0, \quad u(x,1) = x - x^2, \tag{7.442}$$

$$u(0,y) = 0, \quad u(1,0) = 0. \tag{7.443}$$

Solution

If we assume that the solution has the form

$$u(x,t) = X(x)Y(y), \tag{7.444}$$

then substituting into (7.441) and dividing by XY we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \tag{7.445}$$

Given that the left hand side is a function of the independent variable x only, and the right hand side of variable y only, it can be concluded that both sides are constant, i.e.

$$\frac{X''}{X} = \lambda = -k^2, \tag{7.446}$$

$$\frac{Y''}{Y} = -\lambda = -k^2. \tag{7.447}$$

From (7.442) and (7.443), for boundary conditions we obtain

$$\begin{aligned} u(x,0) &= X(x)\cdot Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0,\\ u(0,y) &= X(0)\cdot Y(y) = 0 \quad \Rightarrow \quad X(0) = 0,\\ u(1,0) &= X(1)\cdot Y(0) = 0 \quad \Rightarrow \quad X(1) = 0,\\ u(x,1) &= X(x)\cdot Y(1) = x - x^2 \quad \Rightarrow \quad Y(1) = \frac{x - x^2}{X(x)}. \end{aligned} \tag{7.448}$$

From (7.446) we obtain

$$X = c_1 \sin kx + c_2 \cos kx. \tag{7.449}$$

Using the initial conditions (7.448), we obtain

$$X(0) = c_2 = 0 \quad \wedge \quad X(1) = c_1 \sin k = 0 \quad \Rightarrow \quad k = n\pi,\ k \in Z^+,$$

and we further obtain

$$X_n = c_1 \sin(n\pi x).$$

From (7.447) we obtain

$$Y(y) = c_3 \cdot e^{ky} + c_4 \cdot e^{-ky}. \tag{7.450}$$

The initial condition is

$$Y(0) = c_3 + c_4 = 0 \quad \Rightarrow \quad c_3 = -c_4,$$

so that

$$Y_n = c_3\left(e^{ky} - e^{-ky}\right) = c_3 2\frac{e^{ky} - e^{-ky}}{2} = 2c_3\,\mathrm{sh}(n\pi y).$$

Now we have

$$u_n = X(x)_n \cdot Y(y)_n \quad \Rightarrow \quad u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \underbrace{2c_1c_3}_{a_n} \sin(n\pi x)\,\mathrm{sh}(n\pi y). \tag{7.451}$$

From the remaining boundary condition it follows

$$u(x,1) = x - x^2 = \sum_{n=1}^{\infty} \underbrace{a_n\,\mathrm{sh}(n\pi)}_{A_n} \sin(n\pi x),$$

i.e.

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) = x - x^2. \tag{7.452}$$

The left hand side represents a Fourier sine series, and thus, by expanding the function at the right hand side, for coefficients A_n we obtain

$$A_n = \frac{2}{1}\int_0^2 (x - x^2)\sin(n\pi x)\,\mathrm{d}x = \frac{16}{n^3\pi^3}(1 - \cos(n\pi)). \tag{7.453}$$

Given that $A_n = a_n\,\mathrm{sh}(n\pi)$, for these coefficients we obtain

$$a_n = \frac{4(1 - (-1)^n)}{n^3\pi^3\,\mathrm{sh}(n\pi)}, \tag{7.454}$$

and the final solution is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{1 - 1(-1)^n}{n^3} \sin(n\pi x)\frac{\mathrm{sh}(n\pi y)}{\mathrm{sh}(n\pi)}. \tag{7.455}$$

7.8.1 Appendix

When solving second order partial differential equation of special importance is the form of function $f(r)$, where $r^2 = x_i x_i$, $i = 1, 2, \ldots, n$, $n > 1$, that satisfies the Laplace equation[16] :

$$\Delta f(r) = 0, \quad \text{or} \quad \delta_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

Given that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^2},$$

[16]The proof of this generalization was proposed and done by prof. J. Jarić.

it follows

$$\frac{\partial f}{\partial x_i} = f'(r)\frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{r}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = f''\frac{x_i x_j}{r^2} + \frac{f'}{r}\left[\delta_{ij} - \frac{x_i x_j}{r^2}\right],$$

$$f'(r) = \frac{\mathrm{d}f}{\mathrm{d}r}; \quad f''(r) = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} \quad \Rightarrow$$

$$\Delta f = f''(r) + \frac{n-1}{r}f'(r) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}f'}{f'} = \frac{-n+1}{r}\mathrm{d}r \quad \Rightarrow$$

$$f'(r) = Cr^{1-n}, \quad C = \text{const.} \quad \Rightarrow \quad \mathrm{d}f = \frac{C}{r^{n-1}}\mathrm{d}r.$$

Analysis

We shall distinguish two cases

1) for $n = 2$

$$\mathrm{d}f = C\frac{\mathrm{d}r}{r} \Rightarrow f = C\ln r + D,$$

i

2) for $n > 2$

$$f = \frac{C}{2-n}r^{2-n} + D.$$

R Note that in this appendix the convention on summation by repeated indices was used.

VI

Fractional Calculus

8. Introduction to the Fractional Calculus

8.1 Brief History of Fractional Calculus

Fractional calculus[1] (FC) is an extension of ordinary calculus with more than 300 years of history. FC is a venerable branch of mathematics, first conceptualised in 1695 in a series of letters. FC was initiated by Leibniz and L'Hospital as a result of a correspondence which lasted several months in 1695. In that year, Leibniz wrote a letter to L'Hospital raising the following question [1]:

"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hopital was somewhat curious about the above question and replied by another simple one to Leibniz: "What if the order will be 1/2?". Leibniz in a letter dated September 30, 1695, replied: "It will lead to a paradox, from which one day useful consequences will be drawn."

That date is regarded as the exact birthday of the fractional calculus. The issue raised by Leibniz for a fractional derivative (semi-derivative, to be more precise) was an ongoing topic in decades to come [1,2]. Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best mathematical minds in Europe. Euler [2],wrote in 1730:

"*When n is a positive integer and p is a function of* $x, p = p(x)$*, the ratio of* $d^n p$ *to* dx^n *can always be expressed algebraically. But what kind of ratio can then be made if n be a fraction¿*'

Subsequent references to fractional derivatives were made by Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, and others [3-5]. During the 19^{th} century, the theory of fractional calculus was developed primarily in this way, trough insight and genius of great mathematicians. Namely, in 1819 Lacroix [6], gave the correct answer to the problem raised by Leibnitz and L'Hospital for the first time, claiming that $d^{1/2}x/dx^{1/2} = 2\sqrt{x/\pi}$. In his 700 pages long book on Calculus published in 1819, Lacroix developed the formula for n-th derivative of $y = x^m$,with m being a

[1]The first chapter of a scientific monograph [10] is used as base for writing this chapter.

positive integer

$$D_x^n y = \frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}, m \geq n \tag{8.1}$$

Replacing the factorial symbol by Gamma function (8.3), he developed the formula for the fractional derivative of a power function

$$D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha} \tag{8.2}$$

where α and β are fractional numbers and where the gamma function $\Gamma(z)$ (see Section 5.6.1, on page 244) is defined for $z > 0$ as:

$$\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \tag{8.3}$$

In particular, Lacroix calculated

$$D_x^{1/2} x = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = 2\sqrt{\frac{x}{\pi}} \tag{8.4}$$

Surprisingly, the previous definition gives **a nonzero value for the fractional derivative of a constant function** $(\beta = 0)$, since

$$D_x^\alpha 1 = D_x^\alpha x^0 = \frac{1}{\Gamma(1-\alpha)}x^{-\alpha} \neq 0 \tag{8.5}$$

Using linearity of fractional derivatives, the method of Lacroix is applicable to any analytic function by term-vise differentiation of its power series expansion. Unfortunately, this class of functions is too narrow in order for the method to be considered general.

It is interesting to note that simultaneously with these initial theoretical developments, first practical applications of fractional calculus can also be found. In a sense, the first of these was the discovery by Abel in 1823,[7-9]. Abel considered the solution of the integral equation related to the **tautochrone problem**[2]. He found that the solution could be accomplished via an integral transform, which could be written as a semi-derivative. More precisely, the integral transform considered by Abel was

$$K = \int_0^x (x-t)^{-1/2} f(t)\,dt, \quad K = const. \tag{8.6}$$

He wrote the right hand side of (8.6) by means of a fractional derivative of order $1/2$,

$$\sqrt{\pi}\left(\frac{d^{-1/2}}{dx^{-1/2}}(f(x))\right) \tag{8.7}$$

Abel's solution had attracted the attention of Joseph Liouville, who made the first major study of fractional calculus,[11-14]. The most critical advances in the subject came around 1832 when he began to study fractional calculus in earnest and then managed to apply his results to problems

[2]The tautochrone problem consists of the determination of a curve in the (x, y) plane such that the time required for a particle to slide down the curve to its lowest point under the influence of gravity is independent of its initial position x_0, y_0 on the curve.

in potential theory. Liouville began his theoretical development using the well-known result for derivatives of integer order n

$$D_x^n e^{ax} = a^n e^{ax}. \tag{8.8}$$

Expression (8.8) can rather easily be formally generalized to the case of non-integer values of n, thus obtaining

$$D_x^\alpha e^{ax} = a^\alpha e^{ax} \tag{8.9}$$

By means of Fourier expansion, a wide family of functions can be composed as a superposition of complex exponentials.

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x), \; Re a_n > 0 \tag{8.10}$$

Again, by invoking linearity of the fractional derivative, Liouville proposed the following expression for evaluating the derivative of order α

$$D_x^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x}. \tag{8.11}$$

Previous (8.11) is known as *the Liouville's first formula* for a fractional derivative,[15,16]. However, this formula cannot be seen as a general definition of fractional derivative for the same reason Lacroix formula could not: because of its relatively narrow scope. In order to overcome this, Liouville labored to produce a second definition. He started with a definite integral (closely related to the gamma function):

$$I = \int_0^\infty u^{\beta-1} e^{-xu} du, \; \beta > 0, \, x > 0. \tag{8.12}$$

and derived what is now referred to as *the second Liouville's formula*

$$D_x^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \; \beta > 0 \tag{8.13}$$

None of previous definitions were found to be suitable for a general definition of a fractional derivative. In the consequent years, a number of similar formulas emerged. Greer [17], for example, derived formulas for the fractional derivatives of trigonometric functions using (8.9) in the form:

$$D_x^\alpha e^{iax} = a^\alpha \left(\cos\frac{\pi\alpha}{2} + i\sin\frac{\pi\alpha}{2}\right)(\cos ax + i\sin ax) \tag{8.14}$$

Joseph Fourier [18] obtained the following integral representations for $f(x)$ and its derivatives

$$D_x^n f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi)\, d\xi \int_{-\infty}^{+\infty} t^n \cos[t(x-\xi) + n\pi/2] dt, \tag{8.15}$$

By formally replacing integer n by an arbitrary real quantity α he obtained

$$D_x^\alpha f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi)\, d\xi \int_{-\infty}^{+\infty} t^\alpha \cos[t(x-\xi) + \alpha\pi/2] dt. \tag{8.16}$$

The definitions of Liouville and Lacroix are not equivalent, which led some critics to conclude that one must be "correct" and the other "wrong" [19]. De Morgan, however, wrote [20] that:

"*Both these systems, then, may very possibly be parts of a more general system.*"

Both Liouville's formula and Lacroix's are in fact special cases of what is now called the Riemann-Liouville definition of fractional calculus, (see below). This involves an arbitrary constant of integration c, which when set to zero yields Lacroix's formula and when set to $-\infty$ yields Liouville's.

Probably the most useful advance in the development of fractional calculus was due to a paper written by G. F. Bernhard Riemann [21] during his student days. Unfortunately, the paper was published only posthumously in 1892. Seeking to generalize a Taylor series in 1853, Riemann derived different definition that involved a definite integral and was applicable to power series with non-integer exponents

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t + \Psi(x). \tag{8.17}$$

In fact, the obtained expression is the most-widely utilized modern definition of fractional integral. Due to the ambiguity in the lower limit of integration c, Riemann added to his definition a "complementary" function $\Psi(x)$ where the present-day definition of fractional integration is without the troublesome complementary function.

In [22] 1880, A. Caley referring to Riemann's paper (1847) [21] he says, "*The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of a complimentary function containing an infinity of arbitrary constants*." The question of the existence of a complimentary function caused much confusion. Liouville and Peacock were led in to errors and Riemann became inextricably entangled in his concept of a complimentary function.

Since neither Riemann nor Liouville solved the problem of the complementary function, it is of historical interest how today's Riemann-Liouville definition was finally deduced. The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin in 1869, [23] where he used Cauchy's integral formula as a starting point to reach differentiation with arbitrary index. A. V. Letnikov [24] extended the idea of Sonin a short time later in 1872, [25]. Both tried to define fractional derivatives by utilizing a closed contour. Starting with Cauchy's integral formula for integer order derivatives, given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt, \tag{8.18}$$

the generalization to the fractional case can be obtained by replacing the factorial with Euler's Gamma function $\alpha! = \Gamma(1+\alpha)$. However, the direct extension to non-integer values α results in the problem that the integrand in (8.18) contains a branching point, where an appropriate contour would then require a branch cut which was not included in the work of Sonin and Letnikov. Finally, Laurent [26], used a contour given as an open circuit (known as *Laurent loop*) instead of a closed circuit used by Sonin and Letnikov and thus produced today's definition of the *Riemann-Liouville fractional integral*

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)\, dt, \quad Re(\alpha) > 0. \tag{8.19}$$

In expression (8.19) one immediately recognizes Riemann's formula (8.17), but without the problematic complementary function. In nowadays terminology, expression (8.19) with lower terminal $c = -\infty$ is referred as Liouville fractional integral; by taking $c = 0$ the expression reduces to the so called Riemann fractional integral, whereas the expression (8.19) with arbitrary lower terminal c is called Riemann-Liouville fractional integral. Expression (8.19) is the most widely utilized definition of the fractional integration operator in use today. By choosing $c = 0$ in (8.19) one obtains the Riemann's formula (8.17) without the problematic complementary function $\Psi(x)$

and by choosing $c = -\infty$, formula (8.19) is equivalent to Liouville's first definition (8.10). These two facts explain why equation (8.19) is called *Riemann-Liouville fractional integral*. While the notation of fractional integration and differentiation only differ in the sign of the parameter α in (8.19), the change from fractional integration to differentiation cannot be achieved directly by inserting negative α at the right-hand side of (8.19). The problem originates from the integral at the right side of (8.19) which is divergent for negative integration orders. However, by analytic continuation it can be shown that

$$D_{c,x}^{\alpha} f(x) = D_{c,x}^{n-\beta} f(x) = D_{c,x}^{n} f(x)\, D_{c,x}^{-\beta} f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\beta)}\int_c^x (x-t)^{\beta-1} f(t)\,dt\right), \tag{8.20}$$

holds, which is known today as the definition of the *Riemann-Liouville fractional derivative*. In (8.20) $n = [\alpha]$ is the smallest integer greater than α with $0 < \beta = n - \alpha < 1$. For either $c = 0$ or $c = \infty$ the integral in (8.20) is the Beta-integral for a wide class of functions and thus easily evaluated. The Riemann-Liouville model can be used to describe processes with power-law behaviour, due to the power-function kernel in the definition of the integral transform, but there are many other types of behaviours that occur in nature and that cannot be described by simple power functions.

Nearly simultaneously, Grunwald and Letnikov provided the basis for another definition of fractional derivative [27] which is also frequently used today. Disturbed by the restrictions of the Liouville's approach Grunwald (1867) adopted the definition of a derivative as the limit of a difference quotient as its starting point. He arrived at definite-integral formulas for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit, and also that the Liouville's definition, in which no distinguishable lower limit appeared, correspond to a lower limit $-\infty$. Formally,

$${}^{GL}D_x^{\alpha} f(x) = \lim_{h\to 0}\frac{\Delta_h^{\alpha} f(x)}{h^{\alpha}} = \lim_{h\to 0}\frac{\sum\limits_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} f(x-kh)}{h^{\alpha}} \quad \alpha > 0 \tag{8.21}$$

which is today called the *Grunwald-Letnikov fractional derivative*. In definition (8.21), $\binom{\alpha}{k}$ is the generalized binomial coefficient, wherein the factorials are replaced by Euler's Gamma function. Letnikov [24] also showed that definition (8.21) coincides, under certain relatively mild conditions, with the definitions given by Riemann and Liouville. Today, the Grunwald-Letnikov definition is mainly used for derivation of various numerical methods, which use formula (8.21) with finite sum to approximate fractional derivatives.

Also, in 1888-1891 Nekrasov [28,29] gave applications of fractional integro-differentiation in the form (8.18) to the integration of high order differential euations.

Together with the advances in fractional calculus at the end of the nineteenth century the work of O. Heaviside [30] has to be mentioned. The operational calculus of Heaviside, developed to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. The connection to fractional calculus has been established by the fact that Heaviside used arbitrary powers of p, mostly $\sqrt{p}$, to obtain solutions of various engineering problems.

Weyl [31] and Hardy, [32,33], also examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes in 1917. Moreover, Weyl showed that the following fractional integrals could be written for $0 < \alpha < 1$ assuming that the integrals in (8.22) are convergent over an infinite interval

$$I_+^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\varphi(t)\,dt, \quad I_-^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\varphi(t)\,dt, \tag{8.22}$$

Specially, the Riemann-Liouville definition of a fractional integral given in (8.19) with lower limit $c = -\infty$, the form equivalent to the definition of fractional integral proposed by Liouville, is also often referred to as *Weyl fractional integral*. In the modern terminology one recognizes two distinct variants of all fractional operators, left sided and right sided ones. Weyl operators defined in (8.22) are sometimes also referred to as the left and right Liouville fractional integrals, respectively.

Later, in 1927 Marchaud [34] developed an integral version of the Grunwald-Letnikov definition (8.21) of fractional derivatives, using

$$^{M}D_{x}^{\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{\left(\Delta_{t}^{l} f\right)(x)}{t^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt, \quad \alpha > 0 \tag{8.23}$$

as fractional derivative of a given function f, today known as *Marchaud fractional derivative*. The term $\left(\Delta_{t}^{l} f\right)(x)$ is a finite difference of order $l > \alpha$ and c is a normalizing constant. Since this definition is related to the Grunwald-Letnikov definition,it also coincides with the Riemann-Liouville definition under certain conditions. M. Riesz published a number of papers starting from 1938 [35,36] which are centered around the integral

$$^{R}I^{\alpha} \varphi = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{\varphi(t)}{|t-x|^{1-\alpha}} dt, \quad Re\alpha > 0, \quad \alpha \neq 1, 3, 5, \ldots \tag{8.24}$$

today known as *Riesz potential*. This integral (and its generalization in the n-dimensional Euclidean space) is tightly connected to Weyl fractional integrals (8.22) and therefore to the Riemann-Liouville fractional integrals by

$$^{R}I^{\alpha} = \left(I_{+}^{\alpha} + I_{-}^{\alpha}\right) \left(2\cos(\alpha\pi/2)\right)^{-1} \tag{8.25}$$

In 1949. Riesz [37] also developed a theory of fractional integration for functions of more than one variable.

A modification of the Riemann-Liouville definition of fractional integrals, given by

$$\frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_{0}^{x} \left(x^{2} - t^{2}\right)^{\alpha-1} t^{2\eta+1} \varphi(t)\, dt, \qquad \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_{0}^{x} \left(x^{2} - t^{2}\right)^{\alpha-1} t^{1-2\alpha-2\eta} \varphi(t)\, dt, \tag{8.26}$$

were introduced by Erdelyi et al. in [38-40], which became useful in various applications. While these ideas are tightly connected to fractional differentiation of the functions $x^2$and $\sqrt{x}$, already done by Liouville 1832, the fact that Erdelyi and Kober used the Mellin's transform for their results is noteworthy.

Among the most significant modern contributions to fractional calculus are those made by the results of M. Caputo in 1967,[41]. One of the main drawbacks of Riemann-Liouville definition of fractional derivative is that fractional differential equations with this kind of differential operator require a rather "strange" set of initial conditions. In particular, values of certain fractional integrals and derivatives need to be specified at the initial time instant in order for the solution of the fractional differential equation to be found. Caputo [41,42] reformulated the more "*classic*" definition of the Riemann-Liouville fractional derivative in order to use classical initial conditions, the same one needed by integer order differential equations [40]. Given a function $f$with an $(n-1)$ absolutely continuous integer order derivatives, Caputo defined a fractional derivative by the following expression

$$D_{*}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} \frac{\mathrm{d}^{n} f(s)}{\mathrm{d}s^{n}} \mathrm{d}s. \tag{8.27}$$

Derivative (8.27) is strongly connected to the Riemann-Liouville fractional derivative and is today frequently used in applications. It is interesting to note that Rabotnov [43,44] introduced the same differential operator into the Russian viscoelastic literature a year before Caputo's paper was published. Namely, the fractional exponential function $\ni_{\alpha}(\beta,t)$ introduced by Rabotnov as (8.28) is known also as the **Rabotnov function** and as a special case of the Mittag-Leffler function widely used in fractional calculus. The corresponding kernel is:

$$\ni_{\alpha}(\beta,t-\tau)=(t-\tau)^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^{n}(t-\tau)^{n(1+\alpha)}}{\Gamma[(n+1)(1+\alpha)]}. \tag{8.28}$$

In the late 20^{th} century, fractional calculus began to undergo a large increase in popularity and research output.

By the second half of the twentieth century, the field of fractional calculus had grown to such extent that in 1974 the first conference "*The First Conference on Fractional Calculus and its Applications*" concerned solely with the theory and applications of fractional calculus was held in New Haven. In the same year, the first book on fractional calculus by Oldham and Spanier [3] was published after a joint collaboration started in 1968. A number of additional books have appeared since then, for example McBride (1979) [45], Nishimoto (1991) [46], Miller and Ross (1993), [4], Samko et al. (1993),[47], Kiryakova (1994) [48], Rubin (1996) [49], Carpinteri and Mainardi (1997),[50], Davison and C. Essex (1998), [51],Podlubny (1999) [52], R. Hilfer (2000) [53], Kilbas et.al (2006),[5], Das (2007)[54], J. Sabatier *et. al* (2007) [55], and others. In 1998 the first issue of the mathematical journal *"Fractional calculus & applied analysis"* was printed. This journal is solely concerned with topics on the theory of fractional calculus and its applications. Finally, in 2004 the first conference *"Fractional differentiation and its applications*" was held in Bordeaux, and it is organized every second year since 2004,[56].

Some conferences dedicated, entirely or partly (with special sessions), to FC during the last decades such as: International Carpathian Control Conference (ICCC 2000,2001...), conference on Non-integer Order Calculus and its Applications is organized every year from 2009(RαRNR), Fractional Calculus Day (FCDay first 2009, 2013, 2015,...), International Conference on Analytic Methods of Analysis and Differential Equations is organized every second year (AMADE 1999, 2001, 2003...), Problems and Applications of Operator (OTHA 2011, 2012,...).

8.2 Basic Definitions of Fractional Order Differintegrals

There are many different forms of fractional operators in use today. "Fractional" here does not mean only fractions but it stands for arbitrary quantity including integers, fractions, general complex numbers. Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl and Erdely-Kober derivatives and integrals are the ones mentioned in the previous historical survey. In addition, most of these operators can be defined in two distinct forms, as the left and as right fractional operators. The three most frequently used definitions for the general fractional differintegral are: the Grunwald-Letnikov (GL), the Caputo(C) and the Riemann-Liouville (RL) definitions,[3-5],[52]. A short account of these most frequently used operators is given next.

For expression of the Riemann-Liouville definition, we will consider the Riemann-Liouville n-fold integral for $n\in\mathbb{N}$, $n>0$ defined as (this expression is usually referred to as the **Cauchy repeated integration formula**)

$$\underbrace{\int_a^t\int_a^{t_n}\int_a^{t_{n-1}}\cdots\int_a^{t_3}\int_a^{t_2}}_{n-fold} f(t_1)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_{n-1}\mathrm{d}t_n=\frac{1}{\Gamma(n)}\int_a^t(t-\tau)^{n-1}f(\tau)\,\mathrm{d}\tau. \tag{8.29}$$

The **fractional Riemann-Liouville integral** of the order α for the function $f(t)$ for $\alpha, a \in \mathbb{R}$ can be expressed as follows

$$ {}_{RL}I_a^{\alpha} f(t) \equiv {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{8.30} $$

For the case of $0 < \alpha < 1$, $t > 0$, and $f(t)$ being a causal function of t, the fractional integral is presented as

$$ {}_{RL}D_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad 0 < \alpha < 1, \quad t > 0. \tag{8.31} $$

Moreover, the **left Riemann-Liouville fractional integral** and the **right Riemann-Liouville fractional integral** are defined [5],[39],[44]respectively as

$$ {}_{RL}I_a^{\alpha} f(t) \equiv {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{8.32} $$

$$ {}_{RL}I_b^{\alpha} f(t) \equiv {}_{RL}D_{a,b}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{8.33} $$

where $\alpha > 0$, $n-1 < \alpha < n$. Both Gamma function and Riemann-Liouville fractional integral can be defined for an arbitrary complex order α with positive real order, as well as for purely imaginary order α. Here, the operations of only real order are considered. Furthermore, the **left Riemann-Liouville fractional derivative** is defined as

$$ {}_{RL}D_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{8.34} $$

and the **right Riemann-Liouville fractional derivative** is defined as

$$ {}_{RL}D_{t,b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{8.35} $$

where $n-1 \le \alpha < n$, a, b are the terminal points of the interval $[a,b]$, which can also be $(-\infty, \infty)$.

In the case of the $\alpha \in (0,1)$ the left Riemann-Liouville fractional derivative is reduced to

$$ {}_{RL}D_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(\tau)(t-\tau)^{-\alpha}\,\mathrm{d}\tau. \tag{8.36} $$

For integer values of order α the Riemann-Liouville derivative coincides with the classical, integer order one. In particular [57]

$$ \lim_{\alpha \to (n-1)^+} {}_{RL}D_{a,t}^{\alpha} f(t) = \frac{\mathrm{d}^{n-1} f(t)}{\mathrm{d}t^{n-1}} \tag{8.37} $$

and

$$ \lim_{\alpha \to n^-} {}_{RL}D_{a,t}^{\alpha} f(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}. \tag{8.38} $$

Moreover, very interesting property of the RL fractional derivative is that the fractional derivative of a constant C is not equal to zero. The RL fractional derivative of a constant C takes the form

$$ {}_{RL}D_{a,t}^{\alpha}C = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \neq 0. \tag{8.39}$$

However, definitions of the fractional differentiation of Riemann-Liouville type create a conflict between the well-established and polished mathematical theory and proper needs, such as the initial problem of the fractional differential equation, and the nonzero problem related to the Riemann-Liouville derivative of a constant. A solution to this conflict was proposed by Caputo, see [41,42]. **The left Caputo fractional derivative** is

$$ {}_{C}D_{a,t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\mathrm{d}\tau, \quad n-1 \leq \alpha < n \in Z^+ \tag{8.40}$$

and the **right Caputo fractional derivative** is

$$ {}_{C}D_{t,b}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}\frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\mathrm{d}\tau, \quad n-1 \leq \alpha < n \in Z^+. \tag{8.41}$$

It is obvious from the definition (8.40) that the Caputo fractional derivative of a constant is zero. Regarding continuity with respect to the differentiation order, Caputo derivative satisfies the following limits

$$ \lim_{\alpha\to(n-1)^+} {}_{C}D_{a,t}^{\alpha}x(t) = \frac{\mathrm{d}^{n-1}x(t)}{\mathrm{d}t^{n-1}} - D^{(n-1)}x(a) \tag{8.42}$$

and

$$ \lim_{\alpha\to n-} {}_{C}D_{a,t}^{\alpha}x(t) = \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \tag{8.43}$$

Obviously, Riemann-Liouville operator ${}_{RL}D_a^n$, $n \in (-\infty,\infty)$, varies continuously with n. This is not the case with the Caputo derivative. Obviously, Caputo derivative is stricter than Riemann-Liouville derivative; one reason is that the n-th order derivative is required to exist. On the other hand, the initial conditions of fractional differential equations with Caputo derivative have a clear physical meaning and Caputo derivative is extensively used in engineering applications. The left and right Riemann-Liouville and Caputo fractional derivatives are interrelated by the following expressions

$$ {}_{RL}D_{a,t}^{\alpha}f(t) = {}_{C}D_{a,t}^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha} \tag{8.44}$$

$$ {}_{RL}D_{t,b}^{\alpha}f(t) = {}_{C}D_{t,b}^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}. \tag{8.45}$$

Grunwald and Letnikov defined fractional derivative in the following way

$$ \begin{aligned} {}_{GL}D_x^{\alpha}f(x) &= \lim_{h\to 0}\frac{(\Delta_h^{\alpha}f(x))}{h^{\alpha}} \\ \Delta_h^{\alpha}f(x) &= \sum_{0\leq|j|<\infty}(-1)^{|j|}\binom{\alpha}{j}f(x+jh), \quad h>0, \end{aligned} \tag{8.46}$$

known as the **left Grunwald-Letnikov (GL) derivative**. This derivative can be seen as a limit of the fractional order backward difference. The right sided derivative is defined accordingly

$$
\begin{aligned}
{}_{GL}D_x^{\alpha} f(x) &= \lim_{h\to 0} \frac{(\Delta_{-h}^{\alpha} f(x))}{h^{\alpha}} \\
\Delta_{-h}^{\alpha} f(x) &= \sum_{0\le |j|<\infty} (-1)^{|j|} \binom{\alpha}{j} f(x+jh), \quad h>0,
\end{aligned}
\tag{8.47}
$$

Definitions (8.28) and (8.47) are valid for both $\alpha > 0$ (fractional derivative) and for $\alpha < 0$ (fractional integral) and, commonly, these two notions are grouped into one single operator called **GL differintegral**. The GL derivative and RL derivative are equivalent if the functions they act upon are sufficiently smooth. The generalized binomial coefficients, calculation for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$, is the following

$$
\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \quad \binom{\alpha}{0} = 1. \tag{8.48}
$$

Let us consider $n = (x-a)/h$ where a is a real constant. This constant can be interpreted as the lower terminal (an analogue of the lower integration limit, necessary even for the derivative operator due to its non-local properties). The GL differentigral can be expressed as a limit

$$
{}_{GL}D_{a,t}^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{[\frac{x-a}{h}]} (-1)^j \binom{\alpha}{j} f(x-jh) \tag{8.49}
$$

where $[x]$ means the integer part of x, a and t are the bounds of the operation for ${}_{GL}D_{a,t}^{\alpha} f(x)$. For the numerical calculation of fractional-order derivatives we can use the following relation (8.50) derived from the GL definition (8.49). The relation to the explicit numerical approximation of the α-th derivative at the points kh, $(k = 1,2,\dots)$ has the following form, [52]

$$
{}_{x-L}D_x^{\pm\alpha} f(x) \approx h^{\mp\alpha} \sum_{j=0}^{N(x)} b_j^{\pm\alpha} f(x-jh) \tag{8.50}
$$

where L is the "memory length", h is the step size of the calculation,

$$
N(x) = \min\left\{ \left[\frac{x}{h}\right], \left[\frac{L}{h}\right] \right\} \tag{8.51}
$$

$[x]$ is the integer part of x and $b_j^{(\pm\alpha)}$ is the binomial coefficient given by

$$
b_0^{(\pm\alpha)} = 1, \quad b_j^{(\pm\alpha)} = \left(1 - \frac{1\pm\alpha}{j}\right) b_{j-1}^{\mu\alpha} \tag{8.52}
$$

This approach is based on the fact that (for a wide class of functions and assuming all initial conditions are zero) the three most commonly used definitions - GL, RL, and Caputo's - are equivalent,[57].

8.3 Basic Properties of Fractional Order Differintegrals

As stated previously, for a wide class of functions, Grunwald-Letnikov definition of the fractional derivative operator coincides with the Riemann-Liouville definition. Thus, in the present section only Riemann-Liouville and Caputo derivatives will be considered. Also, left-side operators are used primarily. Thus, all of the properties presented next will be accounted for this kind of fractional operators only. Similar properties can be formulated and proven for the right-sided operators accordingly [3-5],[52].

Similar to the classical, integer-order integral, the Riemann-Liouville fractional integral satisfies the *semi-group property*,[5] i.e. for any positive orders α and β

$$ {}_{RL}I^{\alpha}_{t,a}\, {}_{RL}I^{\beta}_{t,a} f(t) = {}_{RL}I^{\beta}_{t,a}\, {}_{RL}I^{\alpha}_{t,a} f(t) = {}_{RL}I^{\alpha+\beta}_{t,a}\ {}_{RL} \tag{8.53} $$

Interestingly, the same does also hold for integer order derivatives, but not for fractional order ones. Fractional derivatives do not commute! Let us introduce the following notation

$$ f^{(n-j)}_{n-\alpha}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-j} {}_{RL}I^{n-\alpha}_{a,t} f(t). \tag{8.54} $$

A combination of Riemann-Liouville derivatives, for example, results in the following expression

$$ {}_{RL}D^{\alpha}_{a,t}\, {}_{RL}D^{\beta}_{a,t} f(t) = {}_{RL}D^{\alpha+\beta}_{a,t} f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\beta}(a)}{\Gamma(1-j-\alpha)}(t-a)^{-j-\alpha} \tag{8.55} $$

with n being the smallest integer bigger then β. Thus, in general,

$$ {}_{RL}D^{\alpha}_{a,t}\, {}_{RL}D^{\beta}_{a,t} f(t) \neq {}_{RL}D^{\beta}_{a,t}\, {}_{RL}D^{\alpha}_{a,t} f(t) \neq {}_{RL}D^{\alpha+\beta}_{a,t} f(t). \tag{8.56} $$

A similar result can be obtained for the Caputo derivative.

It is a well-known fact that the classical derivative is the left inverse of the classical integral. The similar relation holds for the Riemann-Liouville derivative and integral

$$ {}_{RL}D^{\alpha}_{a,t}\, {}_{RL}I^{\alpha}_{a,t} f(t) = f(t). \tag{8.57} $$

The opposite, however, is not true (in both the fractional and integer order case)

$$ {}_{RL}I^{\alpha}_{a,t}\, {}_{RL}D^{\alpha}_{a,t} f(t) = {}_{RL}D^{\alpha+\beta}_{a,t} f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j}. \tag{8.58} $$

Utilizing expression (8.43), similar expressions can be obtained relating the Riemann-Liouville integral and derivative of Caputo type. In particular, assuming that the integrand is continuous or, at least, essentially bounded function, Caputo derivative is also the left inverse of the fractional integral.

It is rather important to notice that the Caputo and the Riemann-Liouville formulations coincide when the initial conditions are zero [52]. Besides, the RL derivative is meaningful under weaker smoothness requirements. In fact, assuming that all initial conditions are zero, a number of relations between the fractional order operators is greatly simplified. In such a case, both fractional integral and fractional derivatives possess the semi-group property; the fractional derivative is both left and right inverse to the fractional integral of the same order; and the operations of fractional integration and differentiation can exchange places freely. In the symbolic notation, for any $0 < \alpha < \beta$

$$ {}_{RL}D^{\alpha}_{a,t}\, {}_{RL}D^{\beta}_{a,t} f(t) = {}_{RL}D^{\beta}_{a,t}\, {}_{RL}D^{\alpha}_{a,t} f(t) = {}_{RL}D^{\alpha+\beta}_{a,t} f(t), \tag{8.59} $$

$$ {}_{RL}I^{\alpha}_{a,t}\, {}_{RL}D^{\alpha}_{a,t} f(t) = {}_{RL}D^{\alpha}_{a,t}\, {}_{RL}I^{\alpha}_{a,t} f(t) = f(t), \tag{8.60} $$

$$ {}_{C}D^{\alpha}_{a,t}\, {}_{C}D^{\beta}_{a,t} f(t) = {}_{C}D^{\beta}_{a,t}\, {}_{C}D^{\alpha}_{a,t} f(t) = {}_{C}D^{\alpha+\beta}_{a,t} f(t), \tag{8.61} $$

$$ {}_{RL}I^{\alpha}_{a,t}\, {}_{C}D^{\alpha}_{a,t} f(t) = {}_{C}D^{\alpha}_{a,t}\, {}_{RL}I^{\alpha}_{a,t} f(t) = f(t). \tag{8.62} $$

Laplace transform is one of the major formal tools of science and engineering, especially when modeling dynamical systems. Also, Laplace transform is also usually used for solving fractional integro-differential equations involved in various engineering problems. The Laplace transform $\mathfrak{L}\{\cdot\}$ of the RL fractional integral (8.32) of $f(t)$ is

$$\mathfrak{L}\{{}_{RL}I_{0,t}^{\alpha} f(t)\} = \frac{1}{s^{\alpha}} F(s). \tag{8.63}$$

Laplace transform of the RL fractional derivative is

$$\mathfrak{L}\{{}_{RL}D_{0,t}^{\alpha} f(t)\} = \int_{0}^{\infty} e^{-st} {}_{RL}D_{0,t}^{\alpha} f(t)\,\mathrm{d}t = s^{\alpha} F(s) - \sum_{k=0}^{n-1} {}_{RL}D_{0,t}^{\alpha-k-1} f(t)|_{t=0}. \tag{8.64}$$

The terms appearing in the sum on the right hand side of the expression (8.64) involve the initial conditions and these conditions must be specified when solving fractional differential equations. Laplace transform of Caputo fractional derivative is

$$\int_{0}^{\infty} e^{-st} {}_{C}D_{0,t}^{\alpha} f(t)\,\mathrm{d}t = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n \tag{8.65}$$

which implies that all the initial conditions required by a fractional differential equation are presented by a set of only classical integer-order derivatives. Note also that the assumption of zero initial conditions is perfectly sensible when implementing fractional order controllers and filters. However, when attempting to simulate a fractional order system, the effect of initial conditions must be taken into consideration. In such a case, also, the difference between various definitions of fractional operators cannot be neglected. Besides that, the geometric and physical interpretations of fractional integration and fractional differentiation can be found in Podlubny's work,[52].

The fractional integrals and derivatives, defined on a finite interval $[a,b]$ of $\mathbb{R}$, are naturally extended to whole axis $\mathbb{R}$. Namely, we can also define the fractional integrals over unbounded intervals and, as left inverses, the corresponding fractional derivatives.

8.4 Some other types of fractional derivatives

The left and right Liouville-Weyl fractional integrals ${}_{-\infty}D_t^{-\alpha} f(t)$ and ${}_tD_{+\infty}^{-\alpha} f(t)$ of order $\alpha > 0$ on the whole axis $\mathbb{R}$ are defined by

$${}_{-\infty}D_t^{-\alpha} f(t) = {}^{LW}I_{+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{8.66}$$

and

$${}_tD_{+\infty}^{-\alpha} f(t) = {}^{LW}I_{-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{8.67}$$

respectively, where $t \in \mathbb{R}$ and $\alpha > 0$.

8.4.1 Left and right Liouville-Weyl fractional derivatives on the real axis

The **left** and **right Liouville-Weyl fractional derivatives** ${}_{-\infty}D_t^{\alpha} f(t)$ and ${}_tD_{+\infty}^{\alpha} f(t)$ of order α on the whole axis $\mathbb{R}$ are defined by

$${}_{-\infty}D_t^{\alpha} f(t) = {}^{LW}D_{+}^{\alpha} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left({}_{-\infty}D_t^{-(n-\alpha)} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_{-\infty}^{t} (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \right), \tag{8.68}$$

and

$$ {}_tD^{\alpha}_{+\infty}f(t) = {}^{LW}D^{\alpha}_{-}f(t) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_tD^{-(n-\alpha)}_{+\infty}f(t)\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_t^{\infty}(\tau-t)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau\right), \tag{8.69} $$

respectively, where $t \in \mathbb{R}$ and $n = [\alpha]+1$, $\alpha \geq 0$. In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$ {}_{-\infty}D^0_t f(t) = {}_tD^0_{+\infty}f(t) = f(t), \tag{8.70} $$

$$ {}_{-\infty}D^n_t f(t) = f^{(n)}(t) \quad \text{and} \quad {}_tD^n_{+\infty}f(t) = (-1)^n f^{(n)}(t) \tag{8.71} $$

where $f^{(n)}(t)$ denotes the classical (integer) derivative of $f(t)$ of order n. If $0 < \alpha < 1$ and $t \in \mathbb{R}$, then

$$ {}_{-\infty}D^{\alpha}_t f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau\right) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{f(t)-f(t-\tau)}{\tau^{\alpha+1}}\,\mathrm{d}\tau, \tag{8.72} $$

and

$$ {}_tD^n_{+\infty}f(t) = -\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^{\infty}(\tau-t)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau\right) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{f(t)-f(t+\tau)}{\tau^{\alpha+1}}\,\mathrm{d}\tau. \tag{8.73} $$

For the **Liouville-Weyl fractional integrals** we can also state the corresponding *semigroup property* for $\alpha \geq 0$, $\beta \geq 0$

$$ {}^{LW}I^{\alpha}_{+}f(t) \cdot {}^{LW}I^{\beta}_{+}f(t) = {}^{LW}I^{\alpha+\beta}_{+}f(t) \tag{8.74} $$

and

$$ {}^{LW}I^{\alpha}_{-}f(t) \cdot {}^{LW}I^{\beta}_{-}f(t) = {}^{LW}I^{\alpha+\beta}_{-}f(t), \tag{8.75} $$

where, for complementation, we have defined ${}^{LW}I^0_{+} = {}^{LW}I^0_{-} = I$ (**identity operator**) and ${}^{LW}D^0_{+} = {}^{LW}D^0_{-} = I$. In fact, we easily recognize the fundamental property

$$ {}^{LW}D^{\alpha}_{+} \cdot {}^{LW}I^{\alpha}_{+} = (-1)^{n\,LW}D^{\alpha}_{-} \cdot {}^{LW}I^{\alpha}_{-} = I. \tag{8.76} $$

Because of the unbounded intervals of integration, fractional integrals and derivatives of Liouville-Weyl type can be successfully handled via the Fourier transform and the related theory of pseudo-differential operators, that simplifies their treatment.

We assume that the integrals in their definitions are in a proper sense, in order to ensure that the resulting functions can be Fourier transformable in the ordinary or generalized sense. Let

$$ \hat{F}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}f(t)\,\mathrm{d}t, \quad \omega \in \mathbb{R} \tag{8.77} $$

be the *Fourier transform* of a function $f(t)$ of real variable $t \in \mathbb{R}$. Let $f(t)$ be defined on $(-\infty,+\infty)$ and $0 < \alpha < 1$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$ \begin{gathered} \hat{F}\left({}_{-\infty}D^{-\alpha}_t f(t)\right) = (i\omega)^{-\alpha}\hat{F}(\omega), \quad \hat{F}\left({}_{-\infty}D^{+\alpha}_t f(t)\right) = (i\omega)^{+\alpha}\hat{F}(\omega), \\ \hat{F}\left(tD^{-\alpha}_{\infty} f(t)\right) = (-i\omega)^{-\alpha}\hat{F}(\omega), \quad \hat{F}\left({}_tD^{+\alpha}_{\infty} f(t)\right) = (i\omega)^{+\alpha}\hat{F}(\omega) \end{gathered} \tag{8.78} $$

8.4.2 Hilfer fractional derivative

Natural question is if there exists an operator with an additional parameter that has two derivatives RL and Caputo as particular cases. The probably simplest construction of such an operator was suggested by Hilfer in [53]. The so-called *generalized Riemann–Liouville fractional derivative* (nowadays referred to as the **Hilfer fractional derivative**) or order α, $n-1<\alpha\le n\in\mathbb{N}$, and type β, $0\le\beta\le 1$, is defined by the following composition of three operators:

$$(D_a^{\alpha,\beta}f)(t)=\left({}_{RL}I_a^{\beta(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[{}_{RL}I_a^{(1-\beta)(n-\alpha)}f\right]\right)(t) \tag{8.79}$$

In particular, if $n=1$, the previous definition is equivalent with

$$(D_a^{\alpha,\beta}f)(t)=\left({}_{RL}I_a^{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_{RL}I_a^{(1-\beta)(1-\alpha)}f\right]\right)(t). \tag{8.80}$$

The fractional operator $(D_a^{\alpha,\beta}f)$ given by (8.80) was firstly introduced by Hilfer [53].

For $\beta=0$, this operator is reduced to the Riemann–Liouville fractional derivative $\left(D_a^{\alpha,0}\equiv{}_{RL}D_a^{\alpha}\right)$ and the case $\beta=1$ corresponds to the Caputo fractional derivative $\left(D_a^{\alpha,1}\equiv{}_{C}D_a^{\alpha}\right)$. Also, one can observe the following properties of Hilfer fractional derivative:

a) The Hilfer derivative $(D_a^{\alpha,\beta}f)$, can be written as:

$$(D_a^{\alpha,\beta}f)(t)=\left({}_{RL}I_a^{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_{RL}I_a^{1-\gamma}f\right]\right)(t)=\left({}_{RL}I_a^{\gamma-\alpha}D_a^{\gamma}f,\right)(t) \tag{8.81}$$

where $\gamma=\alpha+\beta-\alpha\beta$. The parameter γ satisfies $0<\gamma\le 1$, $\gamma\ge\alpha$, $\gamma>\beta$, $1-\gamma<1-\beta(1-\alpha)$. The $(D_a^{\alpha,b}f)$ derivative is considered as an *interpolator* between the Riemann–Liouville and Caputo derivative since

$$(D_a^{\alpha,b}f)=\begin{cases}{}_{RL}D_a^{\alpha}f, & \beta=0,\\ {}_{C}D_a^{\alpha}f, & \beta=1.\end{cases} \tag{8.82}$$

8.4.3 Marchaud fractional derivative

For a function $f\in C^1[a,b]$, $-\infty<a<b<\infty$, the left Marchaud fractional derivative ${}_{M}D_{a,t}^{\alpha}$, $0<\alpha<1$, is defined and given as an equivalent form of the Riemann–Liouville derivative:

$${}_{M}D_{a,t}^{\alpha}f(t)=\frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}}+\frac{\alpha}{\Gamma(1-\alpha)}\int_a^t\frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau,\quad 0<\alpha<1. \tag{8.83}$$

Also, the right Marchaud fractional derivative is defined as

$${}_{M}D_{b,t}^{\alpha}f(t)=\frac{f(t)}{\Gamma(1-\alpha)(b-t)^{\alpha}}+\frac{\alpha}{\Gamma(1-\alpha)}\int_t^b\frac{f(t)-f(\tau)}{(\tau-t)^{1+\alpha}}\,\mathrm{d}\tau,\quad 0<\alpha<1. \tag{8.84}$$

For a more general class of functions f, the left Marchaud derivative is defined as a limit of restricted Marchaud derivatives, i.e the integrals in previous definitions are assumed to be convergent. Namely,

$${}_{M}D_{a,t}^{\alpha}f(t)=\frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}}+\lim_{\varepsilon\to 0}\psi_{\varepsilon}(t) \tag{8.85}$$

where the function $\psi_{\varepsilon}(t)$ has to be defined separately for $a<t<a+\varepsilon$

$$\psi_{\varepsilon}(t)=\int_a^{t-\varepsilon}\frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau,\quad \varepsilon>0. \tag{8.86}$$

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Appendices

A. Fractional Calculus: A Survey of Useful Formulas

A.1 Introduction

The correspondence between L'Hopital and Leibniz, in 1695, about what might be a derivative of order $\frac{1}{2}$, led to the introduction of a generalisation of integral and derivative operators, known as **Fractionul Calculus** (which in spite of its name covers irrational or even complex integration and differentiation orders).

Many expressions of Fractional Calculus have been published, but such results are scattered over the litera,ture and use different notations. This paper intends to gather systematically some of the most useful formulas for reference purposes.

Section 2. presents the notation used and collects the definition and relevant properties of the main special functions that appear in Fractional Calculus.

Section 3. collects some definitions of one-dimensional fractional integral and derivative operators and some of their properties.

Section 4. is a table of frar:tional derivatives.

Section 5. is a table of Laplace and Fourier transforms.

Section 6. collects solutions of some systems of fractional equations.

Section 7. collects some topics about fractional transfer functions.

Section 8. is an introduction to fractional vector operators.

A.2 Notation and Special Functions

A.2.1 Notation

Floor of $x \in \mathbb{R}$ (largest integer not larger than x) - $\lfloor x \rfloor$.

Ceiling of $x \in \mathbb{R}$ (largest integer not larger than x) - $\lceil x \rceil$

variable of the $\mathscr{Z}$-transform (with some abuse of notation, it can be identified with the advance operator, and its inverse with the delay, operator) - z

Heaviside function - $H(x) = \begin{cases} 1, & \text{if}\, x \geq x_0 \\ 0, & \text{if}\, x < x_0. \end{cases}$

Pringsheim notation of continued fraction (which need not have an infinite number of terms)

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5 + \cdots}}}}} = \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \frac{b_4}{a_4}, \cdots \right] = \left[a_0, \frac{b_k}{a_k} \right] \Big|_{k=1}^{+\infty}$$

Levi-Civita symbol

$$\varepsilon_{\ell mn} = \begin{cases} +1, & \text{if}\, (\ell,m,n) = (1,2,3),\, (3,1,2),\, (2,3,1), \\ -1, & \text{if}\, (\ell,m,n) = (1,3,2),\, (3,2,1),\, (2,1,3), \\ 0, & \text{if}\, \ell = m \vee \ell = n, \vee m = n. \end{cases}$$

A.2.2 Definitions of some Special Functions

Euler's gamma function

$$\Gamma(z) = \begin{cases} \int\limits_0^{+\infty} e^{-y} y^{z-1}\, \mathrm{d}y, & \text{if} \quad \Re(z) > 0, \\ \\ \dfrac{\Gamma(z+n)}{(z)_n}, & \text{if}\, \Re(z) > -n, \quad n \in \mathbb{N} \wedge z \notin \mathbb{Z}_0^- \end{cases} . \tag{A.1}$$

Pochhammer function

$$(\rho)_0 = 1 \quad \textit{and} \quad (\rho)_k = \rho(\rho+1)\cdots(\rho+k-1), \quad k \in \mathbb{N}. \tag{A.2}$$

Combinations of a things, b at a time

$$\binom{a}{b} = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, & \text{if} \quad a, b, a-b \notin \mathbb{Z}^- \\ \\ \dfrac{(-1)^b \Gamma(b-a)}{\Gamma(b+1)\Gamma(-a)}, & \text{if} \quad a \in \mathbb{Z}^- \wedge b \in \mathbb{Z}_0^+ \\ \\ 0, & \text{if} \quad [(b \in \mathbb{Z}^- \vee b-a \in \mathbb{N}) \wedge a \notin \mathbb{Z}^-] \vee (a, b \in \mathbb{Z}^- \wedge |a| > |b|). \end{cases} \tag{A.3}$$

Beta function

$$B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{A.4}$$

Digamma function

$$\psi(x) = \frac{\mathrm{d} \log \Gamma(x)}{\mathrm{d}x} = \frac{1}{\Gamma(x)} \frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x}. \tag{A.5}$$

Error function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^z e^{-t^2}\, \mathrm{d}t, \quad z \in \mathbb{C}. \tag{A.6}$$

Complementary error function

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \quad z \in \mathbb{C} \tag{A.7}$$

Mittag-Leffler function, or one-parameter Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad \Re(\alpha) > 0, \quad z \in \mathbb{C}.$$

Generalized Mittag-Leffler function with two-paramete function

$$E_{\alpha\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \quad z \in \mathbb{C}. \tag{A.8}$$

Generalized Mittag-Leffler function with three-paramete function

$$E_{\alpha\beta}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(\beta+\alpha k)k!}, \quad \alpha, \beta, \rho \in \mathbb{C}, \Re(\alpha) > 0, \quad z \in \mathbb{C}. \tag{A.9}$$

Miller-Ross function

$$\mathscr{E}_z(\nu, a) = \sum_{k=0}^{+\infty} \frac{a^k z^{k+\nu}}{\Gamma(\nu+k+1)} = z^\nu E_{1,\nu+1}(az), \quad z \in \mathbb{C}. \tag{A.10}$$

Hypergeometric function

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = 1 + \sum_{k=1}^{\infty} \left[\frac{z^k}{k!} \prod_{n=0}^{k-1} \frac{(a_1+n)(a_2+n)\cdots(a_p+n)}{(b_1+n)(b_2+n)\cdots(b_q+n)}. \right] \tag{A.11}$$

Bessel functions of the first kind

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{1}{2}x\right)^{2m+\alpha}. \tag{A.12}$$

Modified Bessel functions of the first kind

$$I_\alpha(x) = j^{-\alpha} J_\alpha(jx). \tag{A.13}$$

Bessel functions of the second kind

$$Y_\alpha = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin|(\alpha\pi)}. \tag{A.14}$$

Hermite polynomial

$$H_n(x) = e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-x^2}\right). \tag{A.15}$$

A.2.3 Properties of the Mittag-Leffler functions: special values

$$E_{\alpha,1}(z) = E_\alpha(z), \quad E_1(z) = E_{1,1}(z) = \mathscr{E}_t(0,1) = e^z, \quad E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z),$$
$$E_0(z) = \frac{1}{1-z}, \ |z| < 1 \quad \text{and} \quad E_1(z) = E_{1,1}(z) = \mathscr{E}_t(0,1) = e^z,$$
$$E_2(z^2) = E_{2,1}(z^2) = \mathrm{ch}(\sqrt{z}) \quad \text{and} \quad E_{2,2}(z^2) = \frac{\mathrm{sh}(z)}{z},$$
$$E_3(z) = \frac{1}{2}\left[e^{\sqrt[3]{z}} + 2e^{-\frac{1}{2}\sqrt[3]{z}}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{z}\right)\right],$$
$$E_4(z) = \frac{1}{2}\left[\cos\left(\sqrt[4]{z}\right) + \mathrm{ch}\left(\sqrt[4]{z}\right)\right],$$
$$E_{1,2}(z) = \frac{e^z-1}{z} \quad \text{and} \quad E_{2,2} = \frac{\mathrm{sh}(\sqrt{z})}{\sqrt{z}}.$$

A.2.4 Generalized exponential functions

Let $z, \lambda \in \mathbb{C}$, $\Re(\alpha)$ and $n \in \mathbb{N}$ Then

$$E_\alpha(\lambda z^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k+1)}, \tag{A.16}$$
$$e_\alpha^{\lambda z} = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{(k+1)\alpha-1}}{\Gamma(k\alpha+\alpha)}, \tag{A.17}$$
$$e_{\alpha,n}^{\lambda z} = n!\, z^{\alpha-1} E^{n+1}_{\alpha,(n+1)\alpha}(\lambda z^\alpha). \tag{A.18}$$

Generalized exponential functions - the following properties

$${}^cD^\alpha_{a+} E_\alpha(\lambda(z-a)^\alpha)(x) = \lambda E_\alpha(\lambda(x-a)^\alpha), \tag{A.19}$$
$$\lim_{z\to a+} E_\alpha(\lambda(z-a)^\alpha) = 1, \tag{A.20}$$
$$\lim_{z\to a+}\left[(z-a)^{1-\alpha} e_\alpha^{\lambda(z-a)}\right] = \frac{1}{\Gamma(\alpha)}, \tag{A.21}$$
$$\left(\frac{\partial}{\partial z}\right)^n [E_\alpha(\lambda z^\alpha)] = z^{-n} E_{\alpha,1-n}(\lambda z^\alpha), \tag{A.22}$$
$$\left(\frac{\partial}{\partial z}\right)^n \left[e_\alpha^{\lambda z}\right] = z^{\alpha-n-1} E_{\alpha,\alpha-n}(\lambda z^\alpha), \tag{A.23}$$
$$\left(\frac{\partial}{\partial \lambda}\right)^n [E_\alpha(\lambda z^\alpha)] = n!\, z^{\alpha n} E^{n+1}_{\alpha,\alpha n+1}(\lambda z^\alpha), \tag{A.24}$$
$$\left(\frac{\partial}{\partial \lambda}\right)^n \left[e_\alpha^{\lambda z}\right] = n!\, z^{\alpha n+\alpha-1} E^{n+1}_{\alpha,(n+1)\alpha}(\lambda z^\alpha), \tag{A.25}$$
$$e_{\alpha,n}^{\lambda z} = \frac{1}{n!}\left(\frac{\partial}{\partial \lambda}\right)^n \left[e_\alpha^{\lambda z}\right], \quad z \neq 0. \tag{A.26}$$

The generalized α-exponential functions do not have the index property, that is, in general

$$E_\alpha(\lambda z) E_\alpha(\mu z) \neq E_\alpha((\lambda+\mu)z); \quad e_\alpha^{\lambda z} e_\alpha^{\mu z} \neq e_\alpha^{(\lambda+\mu)z}. \tag{A.27}$$

A.3 Fractional Derivatives and Integrals

Let $\alpha \in \mathbb{C} : \Re(\alpha) \in (n-1, n]$, $n \in \mathbb{N}$, and let $[a,b]$ be a finite interval in $\mathbb{R}$.

A.3.1 Definitions of some unidimensional fractional operators

Riemann-Liouville Left-sided Integral

$$ {}^{RL}I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha}} \mathrm{d}u, \quad x \geq a. \tag{A.28}$$

Riemann-Liouville Right-sided Integral

$$ {}^{RL}I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(u)}{(x-u)^{1-\alpha}} \mathrm{d}u, \quad x \leq b. \tag{A.29}$$

Riemann-Liouville Left-sided Derivative

$$ {}^{RL}D^{\alpha}_{a+} f(x) = D^n \, {}^{RL}I^{n-\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{f(u)}{(x-u)^{1-n+\alpha}} \mathrm{d}u, \quad x \geq a. \tag{A.30}$$

Riemann-Liouville Right-sided Derivative

$$ {}^{RL}D^{\alpha}_{b-} f(x) = D^n \, {}^{RL}I^{n-\alpha}_{b-} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b \frac{f(u)}{(u-x)^{1-n+\alpha}} \mathrm{d}u, \quad x \leq b. \tag{A.31}$$

Caputo Left-sided Derivative

$$ {}^{c}D^{\alpha}_{a+} f(x) = {}^{RL}I^{n-\alpha}_{a+} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{1}{(x-u)^{1-n+\alpha}} \frac{\mathrm{d}^n f(u)}{\mathrm{d}u^n} \mathrm{d}u, \quad x \geq a. \tag{A.32}$$

Caputo Right-sided Derivative

$$ {}^{c}D^{\alpha}_{-} f(x) = (-D)^{n L} I^{n-\alpha}_{-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^\infty \frac{f(u)\,\mathrm{d}u}{(u-x)^{1-n+\alpha}}, \quad x < \infty. \tag{A.33}$$

Left-sided Finite-Difference

$$ \Delta^{\alpha}_{h,a+} f(x) = \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(x-kh), \quad x \geq a. \tag{A.34}$$

Right-sided Finite-Difference

$$ \Delta^{\alpha}_{h,b-} f(x) = \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(x+kh), \quad x \leq b. \tag{A.35}$$

Grünwald-Letnikoff Left-sided Derivative

$$ {}^{GL}D^{\alpha}_{a+} f(x) = \lim_{h \to 0+} \frac{\Delta^{\alpha}_{h,a+} f(x)}{h^\alpha}, \quad x \geq a. \tag{A.36}$$

Grünwald-Letnikoff Right-sided Derivative

$$ {}^{GL}D^{\alpha}_{b-} f(x) = \lim_{h \to 0+} \frac{\Delta^{\alpha}_{h,b-} f(x)}{h^\alpha}, \quad x \leq b. $$

Liouville integrals and derivatives are Riemann-Liouville integrals and derivatives for the particular case $a = -\infty$ or $b = +\infty$; that is to say,

$$ {}^{L}I^{\alpha}_{\pm} f(x) \stackrel{def}{=} {}^{RL}I^{\alpha}_{\mp\infty\pm} f(x) \quad \text{and} \quad {}^{L}D^{\alpha}_{\pm} f(x) \stackrel{def}{=} {}^{RL}D^{\alpha}_{\mp\infty\pm} f(x). $$

Sometimes $LI^{\alpha}-$ is named the **fractional integral** of Weyl, and $LD^{\alpha}-$ the Weyl transform, W^{α}.

If D^{α} is any fractional derivative, the Miller-Ross sequential derivative of order $k\alpha, k \in \mathbb{Z}$ is given by

$$ \mathscr{D}^{\alpha} = D^{\alpha}, \quad \mathscr{D}^{k\alpha} = D^{\alpha}\, \mathscr{D}^{(k-1)\alpha}. \tag{A.37} $$

If $f(t)$ has $\beta : \max\{0, \lfloor \alpha \rfloor\}$ continuous derivatives, and $D^{\beta} f(t)$ is integrable, then:

$$ {}^{RL}D^{\alpha}_{a\pm} f(x) = {}^{GL}D^{\alpha}_{a\pm} f(x), \tag{A.38} $$

$$ {}^{C}D^{\alpha}_{a+} f(x) = \left({}^{RL}D^{\alpha}_{a+} \left[f(u) - \sum_{k=0}^{\lceil \Re(\alpha) \rceil - 1} \frac{f^{(k)}(a)}{k!} (u-a)^k \right] \right)(x), \tag{A.39} $$

$$ {}^{C}D^{\alpha}_{b-} f(x) = \left({}^{RL}D^{\alpha}_{b-} \left[f(u) - \sum_{k=0}^{\lceil \Re(\alpha) \rceil - 1} \frac{f^{(k)}(a)}{k!} (b-u)^k \right] \right)(x). \tag{A.40} $$

Equations (A.39)-((A.40) are sometimes considered as the definitions of Caputo derivatives, since they can be applied to a larger set of functions than (A.32)-(A.33).

Whatever the definition employed,

$$ I^{0} f(x) = D^{0} f(x) = f(x). $$

$$ \begin{aligned} {}^{RL}D^{m}_{a+} f(x) &= {}^{L}D^{m}_{+} f(x) = {}^{c}D^{m}_{a+} f(x) = \\ &= {}^{GL}D^{m}_{a+} f(x) = D^{m} f(x) = \frac{\mathrm{d}^{m} f(x)}{\mathrm{d}x^{m}}, \end{aligned} \tag{A.41} $$

$$ \begin{aligned} {}^{RL}I^{m}_{b-} f(x) &= {}^{L}D^{m}_{-} f(x) = {}^{c}D^{m}_{b-} f(x) = \\ &= {}^{GL}D^{m}_{b-} f(x) = (-1)^{m} D^{m} f(x) = (-1)^{m} \frac{\mathrm{d}^{m} f(x)}{\mathrm{d}x^{m}}. \end{aligned} \tag{A.42} $$

Some authors do not distinguish the definition employed by means of a superscript (GL, RL, C, L), but use different fonts for the operator instead (D, D, **D**, $\mathfrak{D}$, $\mathscr{D}$). The particular correspondence between fonts and definitions varies. Very often no indication at all is given, save perhaps in the accompanying text, and the reader is presumed to understand from the context which particular definition is intended.

In the literature, several alternative notations for operator D may be found:

$$ D^{\alpha}_{a+} f(x) = (D^{\alpha}_{a+} f)(x) = {}_{a}D^{\alpha}_{x} f(x) = {}_{a}I^{-\alpha}_{x} f(x) = D^{\alpha}_{x-a} f(x) = \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}(x-a)^{\alpha}} \tag{A.43} $$

$$ D^{\alpha}_{b-} f(x) = (D^{\alpha}_{b-} f)(x) = {}_{x}D^{\alpha}_{b} f(x) = {}_{x}I^{-\alpha}_{b} f(x) = D^{\alpha}_{b-x} f(x) = \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}(b-x)^{\alpha}} \tag{A.44} $$

Only one of the two operators I and D needs to be used, since it is all a matter of changing the sign of α. In practice D is the one more often used.

A.3.2 Properties

Semigroup properties

For $\Re(\alpha) \in (n-1, n]$, $\Re(\beta) \in (m-1, m]$, $m, n \in \mathbb{N}$, and for suitable functions, we have:

$$ {}^{RL}I^{\alpha}_{a+}\,{}^{RL}I^{\beta}_{a+} f(x) = {}^{RL}I^{\alpha+\beta}_{a+} f(x) \quad \text{and} \quad {}^{RL}I^{\alpha}_{b-}\,{}^{RL}I^{\beta}_{b-}\,{}^{RL} f(x) = {}^{RL}I^{\alpha+\beta}_{b-} f(x). \tag{A.45} $$

$$ {}^{RL}D^{\alpha}_{a+}\,{}^{RL}D^{\beta}_{a+}\,{}^{RL} f(x) = {}^{RL}D^{\alpha+\beta}_{a+} f(x) - \sum_{j=1}^{m} {}^{RL}D^{\beta-j}_{a+} f(a) \frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)} \tag{A.46} $$

$$ {}^{RL}D^{\beta}_{a+}\,{}^{RL}I^{\alpha}_{a+} f(x) = {}^{RL}I^{\alpha-\beta}_{a+} f(x) \quad \text{and} \quad {}^{RL}D^{\beta}_{b-}\,{}^{RL}I^{\alpha}_{b-} f(x) = {}^{RL}I^{\alpha-\beta}_{b-} f(x). \tag{A.47} $$

$$ {}^{RL}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+} f(x) = {}^{RL}D^{\alpha}_{b-}\,{}^{RL}I^{\alpha}_{b-} f(x) = f(x.) \tag{A.48} $$

$$ {}^{RL}I^{\alpha}_{a+}\,{}^{RL}D^{\alpha}_{a+} f(x) = f(x) - \sum_{j=1}^{n} \frac{\left({}^{RL}I^{n-\alpha}_{a+} f\right)^{(n-j)}(a)}{\Gamma(\alpha-j+1)} (x-a)^{\alpha-j} \tag{A.49} $$

$$ {}^{RL}I^{\alpha}_{b-}\,{}^{RL}D^{\alpha}_{b-} f(x) = f(x) - \sum_{j=1}^{n} \frac{(-1)^n \left({}^{RL}I^{n-\alpha}_{b-} f\right)^{(n-j)}(a)}{\Gamma(\alpha-j+1)} (b-x)^{\alpha-j}. \tag{A.50} $$

$$ {}^{RL}D^{m\,RL}D^{\alpha}_{a+} f(x) = {}^{RL}D^{\alpha+m}_{a+} f(x), \quad m \in \mathbb{N}. \tag{A.51} $$

$$ {}^{RL}D^{m\,RL}D^{\alpha}_{b-} f(x) = (-1)^{m\,RL} D^{\alpha+m}_{b-} f(x), \quad m \in \mathbb{N}. \tag{A.52} $$

$$ {}^{C}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+} f(x) = {}^{C}D^{\alpha}_{b-}\,{}^{RL}I^{\alpha}_{b-} f(x) = f(x), \quad \Re(\alpha) \notin \mathbb{N} \vee \alpha \in \mathbb{N}. \tag{A.53} $$

$$ {}^{C}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+} f(x) = f(x) - \frac{1}{\Gamma(n-\alpha)}\,{}^{RL}I^{\alpha+1-n}_{a+} f(a)\,(x-a)^{n-\alpha}, \quad \Re(\alpha) \in \mathbb{N} \wedge \Im(\alpha) \neq 0. \tag{A.54} $$

$$ {}^{C}D^{\alpha}_{b-}\,{}^{RL}I^{\alpha}_{a+} f(x) = f(x) - \frac{1}{\Gamma(n-\alpha)}\,{}^{RL}I^{\alpha+1-n}_{a+} f(b)\,(b-x)^{n-\alpha}, \quad \Re(\alpha) \in \mathbb{N} \wedge \Im(\alpha) \neq 0. \tag{A.55} $$

$$ {}^{RL}I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k. \tag{A.56} $$

$$ {}^{RL}I^{\alpha}_{b-}\,{}^{C}D^{\alpha}_{b-} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!} (b-x)^k. \tag{A.57} $$

$$ {}^{L}I^{\alpha}_{+}\,{}^{L}I^{\beta}_{+} f(x) = {}^{L}I^{\alpha+\beta}_{+} f(x) \quad \text{and} \quad {}^{L}I^{\alpha}_{-}\,{}^{L}I^{\beta}_{-} f(x) = {}^{L}I^{\alpha+\beta}_{-} f(x) \tag{A.58} $$

$$ {}^{L}D^{\alpha}_{+}\,{}^{L}I^{\alpha}_{+} f(x) = f(x) \quad \text{and} \quad {}^{L}D^{\alpha}_{-}\,{}^{L}I^{\beta}_{-} f(x) = f(x). \tag{A.59} $$

$$ {}^{L}D_{+}^{\beta}\,{}^{L}I_{+}^{\alpha}f(x) = {}^{L}I_{0+}^{\alpha-\beta}f(x) \quad \text{and} \quad {}^{L}D_{-}^{\beta}\,{}^{L}I_{-}^{\alpha}f(x) = {}^{L}I_{-}^{\alpha-\beta}f(x). \tag{A.60} $$

$$ D^{m}\,{}^{L}I_{+}^{\alpha}f(x) = {}^{L}I_{+}^{\alpha-m}f(x) \quad \text{and} \quad D^{m}\,{}^{L}I_{-}^{\alpha}f(x) = (-1)^{k}\,{}^{L}I_{-}^{\alpha-m}f(x), \quad \Re(\alpha) > m. \tag{A.61} $$

$$ D^{m}\,{}^{L}D_{+}^{\alpha}f(x) = {}^{L}D_{+}^{\alpha+m}f(x) \quad \text{and} \quad D^{m}\,{}^{L}D_{-}^{\alpha}f(x) = (-1)^{k}\,{}^{L}D_{-}^{\alpha+m}f(x). \tag{A.62} $$

Integration by parts

If $\Re(\alpha) > 0$, for suitable functions, we have the following properties

$$ \int_a^b f(x)\left({}^{RL}I_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_a^b f(x)\left({}^{RL}I_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.63} $$

$$ \int_a^b f(x)\left({}^{RL}D_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_a^b g(x)\left({}^{RL}D_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.64} $$

$$ \int_{-\infty}^{\infty} f(x)\left({}^{L}I_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\left({}^{L}I_{-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.65} $$

$$ \int_{-\infty}^{\infty} f(x)\left({}^{L}D_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\left({}^{L}D_{-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.66} $$

$$ \int_{-\infty}^{0} f(x)\left({}^{L}I_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{-\infty}^{0} g(x)\left({}^{L}I_{-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.67} $$

$$ \int_{0}^{\infty} f(x)\left({}^{L}D_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{0}^{\infty} g(x)\left({}^{L}D_{-}^{\alpha}f\right)(x)\,\mathrm{d}x. \tag{A.68} $$

$$ \begin{aligned} \int_a^b f(x)\left({}^{C}D_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x &= \int_a^b g(x)\left({}^{C}D_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x + \\ &+ \left[f(x)\left({}^{RL}I_{a+}^{\alpha}g\right)(x)\right] - \left[g(x)\left({}^{RL}I_{b-}^{\alpha}f\right)(x)\right]. \end{aligned} \tag{A.69} $$

Leibniz formula and derivative of the composition of two functions

$$ \left[{}^{RL}D_{a+}^{\alpha}(fg)\right](x) = \sum_{j=0}^{\infty}\binom{\alpha}{j}\,{}^{RL}D_{a+}^{\alpha-j}f(x)(D^{j}g)(x). \tag{A.70} $$

$$ \begin{aligned} \left[{}^{RL}D_{a+}^{\alpha}(f(g))\right](x) &= \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}f(g(x)) + \\ + \sum_{j=0}^{\infty}\binom{\alpha}{j}\frac{j!\,(x-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}\sum_{i=1}^{j}\left[D^{j}f(g)\right](x)&\sum\prod_{r=1}^{j}\frac{1}{a_r!}\left(\frac{(D^{r}g)(x)}{r!}\right)^{a_r} \end{aligned} \tag{A.71} $$

Taylor's Formula / Serie	The remainder
$f(x)=\sum_{j=0}^{m-1}\frac{{}^{RL}D_{a+}^{\alpha+j}f(x_0)}{\Gamma(\alpha+j+1)}(x-x_0)^{\alpha+j}+R_m(x),\ \alpha>0$	$R_m(x)={}^{RL}I_{a+}^{\alpha+m}\,{}^{RL}I_{a+}^{\alpha+m}f(x)$
$f(x)=\sum_{j=0}^{m}\frac{\Gamma(\alpha)c_j(x_0)}{\Gamma((j+1)\alpha)}(x-x_0)^{(j+1)\alpha-1}+R_m(x)$	$R_m(x)=\frac{({}^{RL}D_{a+}^{\alpha})^{(m+1)x}f(\xi)}{\Gamma((m+1)\alpha+1)}(x-a)^{(m+1)\alpha}$
$f(x)=f(a)+\frac{{}^{C}D_{a+}^{\alpha}f(a)}{\Gamma(\alpha+1)}(x-a)^{\alpha}+\frac{{}^{C}D_{a+}^{\alpha}\,{}^{C}D_{a+}^{\alpha}f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha}+\cdots$	
$f(x)=\sum_{k=0}^{m-1}a_k x^{\alpha_k}+R_m(x),$ where $x>0$ and $a_k=\frac{D^{(\alpha_k)}f(0)}{\Gamma(\alpha_k+1)}$	$R_m(x)=\frac{1}{\Gamma(\alpha_m+1)}\int_0^x(x-z)^{\alpha_m-1}D^{(\alpha_k)}f(z)\,dz$

Table A.1: Fractional Taylor Formulas.

A.3.3 Fractional Taylor Formulas

Among the different generalizations to the fractional case of Taylor series in the literature, we present the following:

In these relations are:

$$\alpha\in[0,1],\quad c_j(x)=(x-x_0)^{1-\alpha}\left[{}^{RL}D_{a+}^{\alpha}f\right]^{j}(x),\quad \xi\in[a,x].$$

In (83), $\alpha_0=0$ and the α_k, $(k=1,\dots,m)$ are an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}$ and

A.4 Analytical Expressions of Some Fractional Derivatives

function - $f(x), x>a$	Fractional Derivative - ${}^{RL}D_{a+}^{\alpha}f(x)$
k	$\frac{k(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$
$(x-a)^{\beta},\quad \Re(\beta)>-1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}$
$e^{\lambda x},\quad \lambda\neq 0$	$e^{\lambda a}(x-a)^{-\alpha}E_{1,1-\lambda}(\lambda(x-a))=e^{\lambda a}\mathscr{E}_{x-a}(-\alpha,\lambda)$
$(x\pm p)^{\lambda},\quad a\pm p>0$	$\frac{(a\pm p)^{\lambda}}{\Gamma(1-\alpha)}(x-a)^{-\alpha}\,{}_2F_1\left(1,-\lambda,1-\alpha;\,\frac{a-x}{a\pm p}\right)$
$(x-a)^{\beta}(x\pm p)^{\lambda},\quad \Re(\beta)>-1\wedge a\pm p>0$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(a\pm p)^{\lambda}(x-a)^{\beta-\alpha}\,{}_2F_1\left(\beta+1,-\lambda;\,\beta-\alpha;\,\frac{a-x}{a\pm p}\right)$
$(x-a)^{\beta}(p-x)^{\lambda},\quad p>x>a\wedge\Re(\beta)>-1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(p-a)^{\lambda}(x-a)^{\beta-\alpha}\,{}_2F_1\left(\beta+1,-\lambda;\,\beta-\alpha;\,\frac{x-a}{p-a}\right)$
$(x-a)^{\beta}e^{\lambda x},\quad \Re(\beta)>-1$	$\frac{\Gamma(\beta+1)e^{\lambda a}}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}\,{}_1F_1\left(\beta+1,\beta+1-\alpha;\,\lambda(x-a)\right)$
$\sin(\lambda(x-a))$	$\frac{(x-a)^{-\alpha}}{2i\Gamma(1-\alpha)}\left[{}_1F_1\left(1,1-\alpha;\,i\lambda(x-a)\right)-{}_1F_1\left(1,1-\alpha;\,-i\lambda(x-a)\right)\right]$
$\cos(\lambda(x-a))$	$\frac{(x-a)^{-\alpha}}{2\Gamma(1-\alpha)}\left[{}_1F_1\left(1,1-\alpha;\,i\lambda(x-a)\right)+{}_1F_1\left(1,1-\alpha;\,-i\lambda(x-a)\right)\right]$
$(x-a)^{\beta}\sin(\lambda(x-a)),\quad \Re(\beta)>-2$	$\frac{\Gamma(\beta+1)}{2i\Gamma(\beta+1-\alpha)},\left[{}_1F_1\left(\beta,\beta-\alpha;\,i\lambda(x-a)\right)-{}_1F_1\left(\beta,\beta-\alpha;\,-i\lambda(x-a)\right)\right](x-a)^{\beta-\alpha}$
$(x-a)^{\beta}\cos(\lambda(x-a)),\quad \Re(\beta)>-1$	$\frac{\Gamma(\beta+1)}{2\Gamma(\beta+1-\alpha)},\left[{}_1F_1\left(\beta,\beta-\alpha;\,i\lambda(x-a)\right)+{}_1F_1\left(\beta,\beta-\alpha;\,-i\lambda(x-a)\right)\right](x-a)^{\beta-\alpha}$
$(x-a)^{\beta}\ln(x-a),\quad \Re(\beta)>-1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}\left[\ln(x-a)+\psi(\beta+1)-\psi(\beta+1-\alpha)\right]$
$(x-a)^{\beta-1}E_{\mu,\beta}((x-a)^{\mu}),\quad \Re(\beta)>0\wedge\Re(\mu)>0$	$(x-a)^{\beta-1-\alpha}E_{\mu,\beta-\alpha}((x-a)^{\mu})$

Table A.2: Analytical Expressions of Some Fractional Derivatives.

R Riemann-Liouville derivatives can be formulated restoring to generalised functions [9], in which case the following additional results can be established:

$$
{}^{RL}D_{a+}^{\alpha}H(x-p)=\begin{cases}\dfrac{(x-\max\{a,p\})^{-\alpha}}{\Gamma(1-\alpha)}, & \text{if } x>p\\[2ex] 0, & \text{if } a\leq x\leq p.\end{cases}
\tag{A.72}
$$

function - $f(x), x > a$	Fractional Derivative - ${}^{RL}D^{\alpha}_{a+}f(x), \quad \Re(\alpha) \geq 0$
$(p-x)^{-\frac{1}{2}}, \; p > x$	$\sqrt{\frac{a}{\pi x}}\frac{1}{p-x} \quad$ for $\; a = 0 \wedge \alpha = \frac{1}{2}$
$x e^{\mu x}$	$x\mathscr{E}_x(-\alpha,\mu) + \alpha\,\mathscr{E}_x(1-\alpha,\mu), \quad$ for $\; a = 0$
$\mathscr{E}_x(\mu,\nu), \quad \mu > -1$	$\mathscr{E}_x(\mu-\alpha,\nu), \quad$ for $\; a = 0$
$x^{\lambda}, \mathscr{E}_x(\mu,\nu), \quad \lambda+\mu > -1$	$\frac{\Gamma(\lambda+\mu+1)x^{\lambda+\mu-\alpha}}{\Gamma(\mu+1)\Gamma(\lambda+\mu+1-\alpha)}\,{}_2F_2(\lambda+\mu+1,\mu+1,\lambda+\mu-\alpha+1;\nu x)$ for $\; a = 0$
$x\mathscr{E}_x(\mu,\nu), \quad \mu > -2$	$x\mathscr{E}_x(\mu-\alpha,\nu) + \alpha\,\mathscr{E}_x(\mu-\alpha+1,\nu), \quad$ for $\; a = 0$
$(x-a)^{-\alpha-1}\sin(2\lambda(x-a))$	$\sqrt{\pi}\,(\frac{x-a}{2\lambda})^{-(\alpha+\frac{1}{2})}\sin(\lambda(x-a))\,J_{-(\alpha+\frac{1}{2})}(\lambda(x-a))$
$(x-a)^{-\alpha-1}\cos(2\lambda(x-a))$	$\sqrt{\pi}\,(\frac{x-a}{2\lambda})^{-(\alpha+\frac{1}{2})}\cos(\lambda(x-a))\,J_{-(\alpha+\frac{1}{2})}(\lambda(x-a))$
$e_{\alpha}^{(\lambda(z-a))}(x)$	$\lambda\, e_{\alpha}^{(\lambda(x-a))}$

Table A.3: Analytical Expressions of Some Fractional Derivatives.

function - $f(x), \quad \Re(\lambda) > 0 \wedge \mu > 0$	Fractional Derivative - ${}^{L}D^{\alpha}_{+}f(x)$
$(b-ax)^{\gamma-1} \quad \Re(\gamma-\alpha) < 1 \wedge a > 0 \wedge ax < b$	$\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)a^{-\alpha}}(b-ax)^{\gamma-1-\alpha}, \quad$ for $\Re(\alpha) \geq 0$
$e^{\lambda x}$	$\lambda^{\alpha}e^{\lambda x}$
$\sin(\mu x)$	$\mu^{\alpha}\sin(\mu x + \frac{\pi\alpha}{2}), \quad$ for $\Re(\alpha) > -1$
$\cos(\mu x)$	$\mu^{\alpha}\cos(\mu x + \frac{\pi\alpha}{2}), \quad$ for $\Re(\alpha) > -1$
$e^{\lambda x}\sin(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{\lambda x}\sin(\mu x + \alpha\,\mathrm{arctg}\,\frac{\mu}{\lambda})$
$e^{\lambda x}\cos(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{\lambda x}\cos(\mu x + \alpha\,\mathrm{arctg}\,\frac{\mu}{\lambda})$

Table A.4: Analytical Expressions of Some Fractional Derivatives.

where is $a \in (-\infty,\infty)$.

$$ {}^{RL}D^{\alpha}_{a+}\frac{\mathrm{d}^n\delta(x-p)}{\mathrm{d}x^n} = \begin{cases} \frac{(x-p)^{-\alpha-n-1}}{\Gamma(-\alpha-n)}, & \text{if } x > p \geq a, \\ 0, & \text{if } a \leq x \leq p \vee p < a. \end{cases} \tag{A.73} $$

where is $n \in \mathbb{N}$, $a \in (-\infty,\infty)$.

A.5 Laplace and Fourier Transforms

It is well known that the Laplace and Fourier transforms, for suitable functions, are given by

$$ \mathscr{L}\varphi(s) = (\mathscr{L}\varphi(t))(s) = \hat{\varphi}(s) = \int_0^{+\infty} e^{-st}\varphi(t)\,\mathrm{d}t \tag{A.74} $$

$$ \mathscr{F}\varphi(\varkappa) = (\mathscr{F}\varphi(x))(\varkappa) = \hat{\varphi}(\varkappa) = \int_{-\infty}^{+\infty} e^{i\varkappa x}\varphi(x)\,\mathrm{d}x. \tag{A.75} $$

A.5.1 Some properties

In connection with the fractional operators we have the following properties for $\Re(\alpha) > 0$, $\Re(\alpha) \in (n-1,n]$ and suitable functions

$$ \mathscr{L}\,{}^{RL}D^{\alpha}_{0+}f(s) = s^{\alpha}\,\mathscr{L}f(s) - \sum_{k=0}^{n-1} s^{n-k-1}\,D^{k}\,{}^{RL}I^{n-\alpha}_{0+}f(0). \tag{A.76} $$

function - $f(x)$, $\Re(\lambda)>0\,\mu>0$	Fractional Derivative - ${}^{L}D_{-}^{\alpha}f(x)$ $(\Re(\alpha)\geq 0)$
$(b+ax)^{\gamma-1}$ $\Re(\gamma-\alpha)<1, \lvert\arg(\frac{a}{b})\rvert<\pi$	$\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)a^{-\alpha}}(b+ax)^{\gamma-1-\alpha}$
$e^{-\lambda x}$	$\lambda^{\alpha}e^{-\lambda x}$
$\sin(\mu x)$	$\mu^{\alpha}\sin(\mu x-\frac{\pi\alpha}{2})$
$\cos(\mu x)$	$\mu^{\alpha}\cos(\mu x-\frac{\pi\alpha}{2})$
$e^{-\lambda x}\sin(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{-\lambda x}\sin(\mu x-\alpha\mathrm{arctg}\frac{\mu}{\lambda})$
$e^{-\lambda x}\cos(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{-\lambda x}\cos(\mu x-\alpha\mathrm{arctg}\frac{\mu}{\lambda})$

Table A.5: Analytical Expressions of Some Fractional Derivatives.

$$\mathscr{L}\,{}^{C}D_{0+}^{\alpha}f(s)=s^{\alpha}\mathscr{L}f(s)-\sum_{k=0}^{n-1}s^{n-k-1}D^{k}f(0). \tag{A.77}$$

$$\mathscr{F}\,{}^{L}I_{+}^{\alpha}f(\varkappa)=\frac{\mathscr{F}f(\varkappa)}{(-i\varkappa)^{\alpha}} \quad\text{and}\quad \mathscr{F}\,{}^{L}I_{-}^{\alpha}f(\varkappa)=\frac{\mathscr{F}f(\varkappa)}{(i\varkappa)^{\alpha}}, \quad (0<\Re(\alpha)<1) \tag{A.78}$$

$$\mathscr{F}\,{}^{L}D_{+}^{\alpha}f(\varkappa)=(-i\varkappa)^{\alpha}\mathscr{F}f(\varkappa) \quad\text{and}\quad \mathscr{F}\,{}^{L}D_{-}^{\alpha}f(\varkappa)=(i\varkappa)^{\alpha}\mathscr{F}f(\varkappa), \tag{A.79}$$

where

$$(\mp i\varkappa)^{\alpha}=\lvert\varkappa\rvert^{\alpha}e^{\mp\alpha\pi i\,\mathrm{sign}(\varkappa)/2}$$

A.5.2 Some Laplace transforms

Table A.6: Some Laplace transforms

$\mathscr{L}\{f(t)\}(s)$	$f(t)$
$\frac{k!\,s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}}$	$t^{\alpha k+\beta-1}\frac{\mathrm{d}^{k}E_{\alpha,\beta}(\pm at^{\alpha})}{\mathrm{d}(\pm at^{\alpha})^{k}}$
$\frac{1}{s^{\alpha}-\lambda}$	$e_{\alpha}^{\lambda t}$
$\frac{n!\,s^{\alpha-1}}{(s^{\alpha}-\lambda)^{n+1}}$	$t^{\alpha n}(\frac{\partial}{\partial\lambda})^{n}E_{\alpha}(\lambda t^{\alpha})$
$\frac{n!}{(s^{\alpha}-\lambda)^{n+1}}$	$(\frac{\partial}{\partial\lambda})^{n}e_{\alpha}^{\lambda z}$
$\frac{s^{\alpha-\beta}}{s^{\alpha}\mp a}$	$t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})$
$\frac{s^{\alpha-1}}{s^{\alpha}\mp a}$	$E_{\alpha}(\pm at^{\alpha})$
$\frac{1}{s^{\alpha}\mp a}$	$t^{\alpha-1}E_{\alpha,\alpha}(\pm at^{\alpha})$
$\frac{s^{1-\beta}}{s\mp a}$	$t^{\beta-1}E_{1,\beta}(\pm at)=\mathscr{E}_t(\beta-1,\pm a)$
$\frac{1}{s^{\beta}}$	$t^{\beta-1}E_{1,\beta}(0)=\mathscr{E}_t(\beta-1,0)=\frac{t^{\beta-1}}{\Gamma(\beta)}$
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{s\sqrt{s}}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{1}{s^{n}\sqrt{s}}$, (n=1,2,...)	$\frac{2^{n}t^{n-\frac{1}{2}}}{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}$
$\frac{s}{(s-a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^{3}}}(e^{bt}-e^{at})$

Continued on next page

$\mathscr{L}\{f(t)\}(s)$	$f(t)$
$\frac{1}{\sqrt{s+a}}$	$\frac{1}{\sqrt{\pi t}} - a\,e^{a^2 t}\,\mathrm{erfc}(a\sqrt{t})$
$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}} + a\,e^{a^2 t}\,\mathrm{erf}(a\sqrt{t})$
$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}} - \frac{2a}{\sqrt{\pi}} e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\tau^2}\,\mathrm{d}\tau$
$\frac{1}{\sqrt{s}\,(s-a^2)}$	$\frac{1}{a}\,e^{a^2 t}\mathrm{erf}(a\sqrt{t})$
$\frac{1}{\sqrt{s}\,(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}\,e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\tau^2}\,\mathrm{d}\tau$
$\frac{b^2-a^2}{(s-a^2)(\sqrt{s}+b)}$	$e^{a^2 t}[b - a\,\mathrm{erf}(a\sqrt{t})] - b\,e^{b^2 t}\,\mathrm{erfc}(b\sqrt{t})$
$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2 t}\,\mathrm{erfc}(a\sqrt{t})$
$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}\,e^{-at}\,\mathrm{erf}(\sqrt{b-a}\,\sqrt{t})$
$\frac{b^2-a^2}{\sqrt{s}\,(s-a^2)(\sqrt{s}+b)}$	$e^{a^2 t}\,[\frac{b}{a}\mathrm{erf}(a\sqrt{t}) - 1] + e^{b^2 t}\,\mathrm{erfc}(b\sqrt{t})$
$\frac{(1-s)^n}{n+\frac{1}{2}}$	$\frac{n!}{(2n)!\,\sqrt{\pi t}}\,H_{2n}(\sqrt{t})$
$\frac{(1-s)^n}{n+\frac{3}{2}}$	$\frac{n!}{(2n+1)!\,\sqrt{\pi}}\,H_{2n+1}(\sqrt{t})$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s}}$	$ae^{-at}[I_1(at) + I_0(at)]$
$\frac{1}{\sqrt{s+a}\,\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}\,I_0(\frac{a-b}{2}\,t)$
$\frac{\Gamma(k)}{(s+a)^k\,(s+b)^k}$, $(k \geq 0)$	$\sqrt{\pi}\,\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}}\,e^{-\frac{1}{2}(a+b)t}\,I_{k-\frac{1}{2}}(\frac{a-b}{2}t)$
$\frac{1}{(s+a)^{\frac{1}{2}}\,(s+b)^{\frac{3}{2}}}$	$t\,e^{-\frac{1}{2}(a+b)t}[I_0(\frac{a-b}{2}t) + I_1(\frac{a-b}{2}t)]$
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}\,I_1(at)$
$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$, $(k > 0)$	$\frac{k}{t}\,e^{-\frac{1}{2}(a+b)t}\,I_k(\frac{a-b}{2}t)$
$\frac{\sqrt{s+a}+(\sqrt{s})^{-2\nu}}{\sqrt{s}\,\sqrt{s+a}}$, $(\nu > -1)$	$\frac{1}{a^\nu}\,e^{-\frac{1}{2}}\,I_\nu(\frac{1}{2}t)$
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\frac{1}{\sqrt{s^2-a^2}}$	$I_0(at)$
$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}$, $(\nu > -1)$	$a^\nu\,J_\nu(at)$
$\frac{1}{(s^2+a^2)^k}$ $(k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\,(\frac{t}{2a})^{k-\frac{1}{2}}\,J_{k-\frac{1}{2}}(at)$
$(\sqrt{s^2+a^2}-s)^k$ $(k > 0)$	$\frac{ka^k}{t}\,J_k(at)$
$\frac{(\sqrt{s^2+a^2}+s)^\nu}{\sqrt{s^2-a^2}}$, $(\nu > -1)$	$a^\nu\,I_\nu(at)$
$\frac{1}{(s^2-a^2)^k}$, (k>0)	$\frac{\sqrt{\pi}}{\Gamma(k)}\,(\frac{t}{2a})^{k-\frac{1}{2}}\,I_{k-\frac{1}{2}}\,(at)$
$\frac{1}{s\sqrt{s+1}}$	$\mathrm{erf}(\sqrt{t})$
$\frac{1}{s+\sqrt{s^2+a^2}}$	$\frac{J_1(at)}{at}$
$\frac{1}{(s+\sqrt{s^2+a^2})^n}$, $n \in \mathbb{N}$	$\frac{nJ_n(at)}{a^n t}$
$\frac{1}{(\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})}$, $n \in \mathbb{N}$	$\frac{J_1(at)}{a}$
$\frac{1}{(\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})^n}$, $n \in \mathbb{N}$	$\frac{J_n(at)}{a^n}$
$\frac{k}{s^2+k^2}\,\mathrm{ch}\,\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$
$\frac{1}{s}\,e^{-k/s}$	$J_0(2\sqrt{kt})$
$\frac{1}{\sqrt{s}}\,e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\,\cos 2\sqrt{kt}$
$\frac{1}{\sqrt{s}}\,e^{k/s}$	$\frac{1}{\sqrt{\pi t}}\,\mathrm{ch}\,2\sqrt{kt}$
$\frac{1}{s\sqrt{s}}\,e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\,\sin 2\sqrt{kt}$

Continued on next page

$\mathscr{L}\{f(t)\}(s)$	$f(t)$
$\frac{1}{s\sqrt{s}}\, e^{k/s}$	$\frac{1}{\sqrt{\pi t}}\, \mathrm{sh}\, 2\sqrt{kt}$
$\frac{1}{s^\nu} e^{-k/s}, \quad (\nu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\nu-1}{2}} J_{\nu-1}\left(2\sqrt{kt}\right)$
$\frac{1}{s^\nu} e^{k/s}, \quad (\nu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\nu-1}{2}} I_{\nu-1}\left(2\sqrt{kt}\right)$
$e^{-k\sqrt{s}}, \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}}\, e^{-\frac{k^2}{4t}}$
$\frac{1}{s}\, e^{-k\sqrt{s}}, \quad (k \geq 0)$	$\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
$\frac{1}{\sqrt{s}}\, e^{-k\sqrt{s}}, \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}}\, e^{-\frac{k^2}{4t}}$
$\frac{1}{s\sqrt{s}}\, e^{-k\sqrt{s}}, \quad (k \geq 0)$	$2\sqrt{\frac{t}{\pi}}\, e^{-\frac{k^2}{4t}} - k\, \mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
$\frac{a\, e^{-k\sqrt{s}}}{s(a+\sqrt{s})}, \ (k \geq 0)$	$-e^{ak}\, e^{a^2 t}\, \mathrm{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}, \ (k \geq 0)$	$e^{ak}\, e^{a^2 t}\, \mathrm{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$
$\frac{s^{\alpha p-\beta}}{(s^\alpha-\lambda)^p}, \ (\Re(\beta) > 0\, \lvert\lambda\, s^{-\alpha}\rvert < 1)$	$t^{\beta-1}\, E^p_{\alpha,\beta}(\lambda\, t^\alpha)$
$\frac{1}{(s^\alpha-\lambda)}, \ (\lvert\lambda\, s^{-\alpha}\rvert < 1)$	$e^{\lambda t}_\alpha$
$\frac{n!}{(s^\alpha-\lambda)^{n+1}}, \ (\lvert\lambda\, s^{-\alpha}\rvert < 1)$ $\frac{n!}{(s^\alpha-\lambda)^{n+1}}, \ (\lvert\lambda\, s^{-\alpha}\rvert < 1)$	$\left(\frac{\partial}{\partial\lambda}\right)^n e^{\lambda t}_\alpha$ $t^{\alpha n}\, e^{\lambda z}_{\alpha,n}$

A.6 Systems of Fractional Equations

Bellow we present the explicit solutions of the Riemann-Liouville and Caputo systems of linear fractional differential equations involving the following matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \bar{\mathbf{B}}(x) = \begin{bmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{bmatrix} \tag{A.80}$$

These kind of systems are a important tool in many applied ar the application to the state-variable technique in control theory.

Problem	Solution
${}^{RL}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\,\bar{Y}(x)$	$\bar{Y}_h(x) = e^{\mathbf{A}(x-a)}_\alpha\, \bar{\mathbf{C}}$, where $\bar{\mathbf{C}}$ is a constant matrix
${}^{RL}D^\alpha_{a+}Y(x) = \mathbf{A}\,Y(x), \ \bar{Y}(x_0) = \bar{Y}_0, \ (x_0 > a)$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha \left(e^{\mathbf{A}(x_0-a)}_\alpha\right)^{-1} \bar{Y}_0$
${}^{RL}D^\alpha_{a+}Y(x) = \mathbf{A}\,Y(x) \ \lim_{x\to a+}[(x-a)^{1-\alpha}\,\bar{Y}(x)] = \bar{Y}_0$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha\, \bar{Y}_0$
${}^{RL}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\,\bar{Y}(x) + \mathbf{B}\,\bar{Y}(x)$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha\, \bar{\mathbf{C}} + \int_a^x e^{\mathbf{A}(x-\xi)}_\alpha\, \bar{\mathbf{B}}(\xi)\, d\xi$
${}^{C}D^\alpha_{a+}Y(x) = \mathbf{A}\,Y(x), \ \bar{Y}(a) = \bar{b}, \ (\bar{b} \in \mathbb{R}^n)$	$\bar{Y}(x) = E_\alpha(A(x-a)^\alpha)\,\bar{b}$
${}^{C}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\,\bar{Y}(x) + \mathbf{B}\,\bar{Y}(x)$	$Y(x) = E_\alpha(A(x-a)^\alpha)\,C + \int_a^x e^{\mathbf{A}(x-\xi)}_\alpha\, \bar{\mathbf{B}}(\xi)\, d\xi$

Table A.7: Systems of Fractional Equations.

where

$$E_\alpha(\mathbf{A}z) = \sum_{k=0}^{infty} \mathbf{A}^k \frac{z^{\alpha k}}{\Gamma(k\alpha+1)} \quad \text{and} \quad E_\alpha(\mathbf{A}z) = \sum_{k=0}^{infty} \mathbf{A}^k \frac{z^{\alpha k}}{\Gamma(k\alpha+1)} \tag{A.81}$$

are, respectively, the natural matrix generalizations of the above mentioned Riemann-Liouville and Caputo σ-exponential.

As to the issue of numerically solving fractional differential equations see [25].

A.7 Transfer Functions

A.7.1 Discrete transfer function approximations

These approximations are discrete transfer functions, i.e. transfer functions that depend on z^{-1}, the inverse of the $\mathscr{Z}$-transform variable of time, which can be identified with the delay operator. The $\mathscr{Z}$-transform of a function sampled with sampling interval h is

$$\mathscr{Z}\{f(t)\}(z) = \sum_{k=0}^{+\infty} z^{-k} f(kh). \tag{A.82}$$

Unless otherwise noted, the approximations below correspond to ${}^{GL}D^{\alpha}_{0+} f(x)$ and are truncated after an arbitrary number N of terms.

Euler or Griinwald-Letnikoff approximation, causal

$$\begin{gathered} {}^{GL}D^{\alpha}_{a+} f(x) \approx \frac{\Delta^{\alpha}_{h,a+}}{h^{\alpha}}, \quad (x > a). \\ \text{Therefore,} \\ s^{\alpha} \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} z^{-k}, \quad (x > a). \end{gathered} \tag{A.83}$$

Euler or Grünwald-Letnikoff approximation, anti-causal

$$\begin{gathered} {}^{GL}D^{\alpha}_{b-} f(x) \approx \frac{\Delta^{\alpha}_{h,b-}}{h^{\alpha}}, \quad (x < b). \\ \text{Therefore,} \\ s^{\alpha} \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} (-1)^k \binom{\alpha}{k} z^{k}, \quad (x < b). \end{gathered} \tag{A.84}$$

Tustin approximation (truncated Maclaurin series)

$$s^{\alpha} \approx \left(\frac{2}{h}\right)^{\alpha} \sum_{k=0}^{N} \sum_{n=0}^{k} \frac{z^{-1} (-1)^n \Gamma(\alpha+1) \Gamma(-\alpha+1)}{\Gamma(\alpha-n+1) \Gamma(n+1) \Gamma(k-n+1) \Gamma(-\alpha-k+n+1)} \tag{A.85}$$

Tustin approximation (truncated continued fraction expansion)

$$s^{\alpha} \approx \left(\frac{2}{h}\right)^{\alpha} \left[1; \frac{2\alpha}{-\frac{1}{z^{-1}} - \alpha}, \frac{\alpha^2 - k^2}{-\frac{2k+1}{z^{-1}}}\right]_{k=1}^{N} \tag{A.86}$$

First-order backwards finite difference approximation (truncated continued fraction expansion)

$$s^{\alpha} \approx \frac{1}{h^{\alpha}} \left[0; \frac{1}{1}, \frac{\alpha z^{-1}}{1}, \frac{-\frac{k(k+\alpha)}{(2k-1)2k} z^{-1}}{1}, \frac{-\frac{k(k-\alpha)}{2k,(2k+1)} z^{-1}}{1}\right]_{k=1}^{N} \tag{A.87}$$

Impulse response approximation

$$s^{\alpha} \approx \frac{h^{-\alpha}}{\Gamma(1-\alpha)} - \frac{h^{-\alpha-1}}{\Gamma(-\alpha)} + \sum_{k=1}^{N} \frac{(kh)^{-\alpha-1}}{\Gamma(-\alpha)} z^{-k} \tag{A.88}$$

Step response approximation

$$s^{\alpha} \approx \frac{h^{-\alpha}}{\Gamma(1-\alpha)} - \frac{h^{-\alpha-1}}{\Gamma(-\alpha)} + \sum_{k=1}^{N} a_k z^{-k}, \quad a_k = \frac{(kh)^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{n=0}^{k-1} a_n, \quad k = 1, 2, \ldots, N. \tag{A.89}$$

(R) In every machine there will be a $k_{\max} \in \mathbb{N}$ which is the largest integer for which $\Gamma(k_{\max})$ does not yet return infinity. Because $\Gamma(x)$ grows very fast, $k_{\max}$ may be relatively small; if $\lfloor \frac{|x-a|}{h} \rfloor > k_{\max}$, the summations in (A.83)-(A.84) will thereby be truncated. To avoid this, the following approximation can be used instead of (A.83), assuming that $k_{\max}$ is even:

$$s^\alpha \approx \frac{1}{h^\alpha} \sum_{k=0}^{k_{\max}} (-1)^k \binom{\alpha}{k} f(t-kh) + \sum_{i=2}^{m} \frac{1}{(ih)^\alpha} \sum_{k=\lceil \frac{k_{\max}}{2} \rceil}^{k_{\max}} (-1)^k \binom{\alpha}{k} f(t-kih) + \\ + \frac{1}{[(m+1)h]^\alpha} \sum_{k=\lceil \frac{k_{\max}}{2} \rceil}^{\lfloor \frac{t-c}{(m+1)h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-k(m+1)h) \quad \text{(A.90)}$$

where

$$\lfloor \frac{t-c}{mh} \rfloor > k_{\max} \geq \lfloor \frac{t-c}{(m+1)h} \rfloor \quad \text{and} \quad m \in \mathbb{N}.$$

The expression (A.84) would be handled in a similar manner.

(R) Each k adds two terms to the continued fraction in (A.87). A truncated MacLaurin series of a first-order backwards finite difference returns the Euler approximation (A.83).

(R) A weighted average of approximations (A.83) and (A.85) is sometimes used [12]. The particular case of weights $\frac{3}{4}$ for (A.83) and $\frac{1}{4}$ for (A.85) is known as the Al-Alaoui operator [13] .

(R) Approximations (A.88)-(A.89) return the exact impulse and step responses at the interval h. From there on, either the impulse or the step response is followed; it is impossible to follow both. For $x = 0$, the output is always far from the exact value.

A.7.2 CRONE or Oustaloup approximation

$$s^\alpha \approx C \prod_{m=1}^{N} \frac{1+\frac{s}{\omega_{z,m}}}{1+\frac{s}{\omega_{p,m}}} \quad \text{where} \quad C = \frac{j\,\omega_C^\alpha}{\prod_{m=1}^{N} \frac{1+\frac{j\omega_C}{\omega_{z,m}}}{1+\frac{j\omega_C}{\omega_{p,m}}}} \quad \text{(A.91)}$$

$$\omega_C \in [\omega_\ell, \omega_h], \quad \omega_{z,m} = \omega_\ell \left(\frac{\omega_h}{\omega_\ell}\right)^{\frac{2m-1-\alpha}{2N}} \quad \omega_{p,m} = \omega_\ell \left(\frac{\omega_h}{\omega_\ell}\right)^{\frac{2m-1+\alpha}{2N}} \quad \text{(A.92)}$$

(R) The N stable real poles and the N stable real zeros of (A.92) are recursively placed it $[\omega_\ell, \omega_h]$, and verify

$$\frac{\omega_{z,m+1}}{\omega_{z,m}} = \frac{\omega_{p,m+1}}{\omega_{p,m}} = \left(\frac{\omega_h}{\omega_\ell}\right)^{\frac{1}{N}} \quad \text{(A.93)}$$

It is advisable to make $N \geq \lfloor \log_{10} \frac{\omega_h}{\omega_\ell} \rfloor$. Typically the approximation will be acceptable in $[10\omega_\ell, \frac{\omega_h}{10}]$. F]equency ω_C, at which the gain will be exact, is arbitrary, but it is reasonable to make $\omega_\ell \ll \omega_C \ll \omega_h$ (e.g. $\omega_C = \sqrt{\omega_\ell\,\omega_h}$). Or, if $1 \in [\omega_\ell, \omega_h]$, calculations can be simplified making

$$\omega_C = 1 \quad \Rightarrow \quad C = \frac{j^\alpha}{\prod_{m=1}^{N} \frac{1+\frac{j}{\omega_{z,m}}}{1+\frac{j}{\omega_{p,m}}}}$$

R CRONE is an acronym of *Commande Robuste d'Ordre Non-Entier*, Flench for Non-Integer Order Robust Control.

A.7.3 Matsuda approximation

Given the frequency behaviour $G(j\omega)$ of transfer function $G(s)$ (which may be fractional), at frequencies $\omega_0, \omega_1, \ldots, \omega_N$ (which do not need to be ordered),

$$G(s) \approx d_0(\omega_0) + \frac{s-\omega_0}{d_1(\omega_1)+} \frac{s-\omega_1}{d_2(\omega_2)+} \frac{s-\omega_2}{d_3(\omega_3)+} \cdots = \left[d_0(\omega_0); \frac{s-\omega_{k-1}}{d_k(\omega_k)} \right]_{k=1}^{N} \tag{A.94}$$

where

$$d_0(\omega) = |G(j\omega)| \quad \text{and} \quad d_k(\omega) = \frac{\omega - \omega_{k-1}}{d_{k-1}(\omega) - d_{k-1}(\omega-1)} \quad (k = 1, 2, \ldots, N).$$

Approximation (A.94) only works if all orders involved are real.

A.7.4 General comments on approximations

R The following applies to ail approximations.

- s^α can be approximated as $\frac{1}{s^{-\alpha}}$ which may be useful one approximation is stable and causal and the other is not.
- can be approximated as $s^\alpha = s^{\lceil \alpha \rceil} s^{\alpha - \lceil \alpha \rceil}$ or as $s^\alpha = s^{\lfloor \alpha \rfloor} s^{\alpha - \lfloor \alpha \rfloor}$, to limit approximation orders to the $[-1, 1]$ range.
- Discrete approximations of s^α can be converted into continuous approximations and continuous ones into discrete ones using the Tustin method or any other such method. Notice that usually continuous-time approximations outperform discrete approximations.
- Transfer function $\frac{b_1 s^{\beta_1} + b_2 s^{\beta_2} + \cdots + b_m s^{\beta_m}}{a_1 s^{\beta_1} + a_2 s^{\beta_2} + \cdots a + m s^{\beta_m}}$ can be approximated finding approximations for $s^{\beta_1}, s^{\beta_2}, \ldots, s^{\beta_m}$ and linearly combining them. But it can also be approximated as a whole, save if the CRONE approximation is used.
- To find more information on the topic consider in the section consult, for instance, [14-18] and the references included in it.

A.8 An Introduction to Fractional Vector Operators

In this section we provide some topics about the fractional multidimensional operators. A first capital contribution was introduced in 1936 by Riesz. He generalized the

Riemann-Liouville integral looking for a solution for some problem in potencial theory in connection with partial differential equations for parabolic and hyperbolic cases. He gave two n-dimensional integral operators which are known as Riesz potencial (see e.g. [19,20,1,10]). The inverse operator of I_α given by Riesz, corresponding to the parabolic case, is usually considered as a Fractional Laplacian. However commonly in the literature the fractional Laplacian is introduced by the following property

$$\left[\mathfrak{F}(-\Delta)^{\frac{1}{2}} f(\mathbf{x})\right](\mathbf{k}) = |\mathbf{k}|^{-\alpha}\,\mathfrak{F} f(\mathbf{k}), \tag{A.95}$$

where $\mathfrak{F}$ is the Fourier transform, but it is well known that there are several operators verifying this condition.

On the other hand, the relevance of vectorial calculus in the scientific fie1d is well known. We gather here some definitions of the extension of this operators to the fractional case.

Let us consider a general vector

$$\mathbf{F}(\mathbf{x}) = F_s(x)\,\mathbf{e}_s = F_x\,\mathbf{e}_x + F_y\,\mathbf{e}_y + F_z\,\mathbf{e}_z, \quad s \in \{1,2,3\}, \tag{A.96}$$

where $\mathbf{e}_s$ are the orthogonal unit vectors. We will use D_s^α to denote any fractional differential operator with respect to the variable x_s.

Gradient of scalar function G	
Classical gradient $= \nabla G$	$\mathrm{grad} G = \frac{\partial G}{\partial x}\mathbf{e}_1 + \frac{\partial G}{\partial y}\mathbf{e}_2 + \frac{\partial G}{\partial z}\mathbf{e}_3$
Fbactional gradient. Definition 1	$\mathrm{grad}^\alpha f(x) = \mathbf{e}_s D_s^\alpha f(x)$
Fbactional gradient. Definition 2	$\mathrm{grad}^\alpha f(x) = \frac{\mathbf{e}_s}{\Gamma(1+\alpha)} D_s^\alpha f(x)$
Divergence of vector function F	
Classical divergence $\nabla\cdot\mathbf{F}(x)$	$\mathrm{div}\mathbf{F}(x) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
Fbactional divergence. Definition 1	$\mathrm{div}^\alpha \mathbf{F}(x) = \mathbf{e}_s D_s^\alpha F_s(x)$
Fbactional divergence. Definition 2	$\mathrm{div}^\alpha \mathbf{F}(x) = \frac{\mathbf{e}_s}{\Gamma(1+\alpha)} D_s^\alpha F_s(x)$
Curl of vector function F	
Classical Curl	$\mathrm{curl}\mathbf{F} = \mathbf{e}_\ell\,\varepsilon_{\ell mn} D_m F_n$
Fbactional Curl. Definition 1	$\mathrm{curl}^\alpha\mathbf{F} = \mathbf{e}_\ell\,\varepsilon_{\ell mn} D_m F_n$
Fbactional Curl. Definition 2	$\mathrm{curl}^\alpha\mathbf{F} = \frac{\mathbf{e}_\ell}{\Gamma(\alpha+1)}\varepsilon_{\ell mn} D_m F_n$
Fbactional Curl. Definition 3	$\mathrm{curl}^\alpha\mathbf{F} = \frac{\mathbf{e}_\ell}{\Gamma(\alpha+1)}\varepsilon_{\ell mn} D_m I_n^{1-\alpha} F_n$

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Index